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A unit of study for modern math was designed to allow for a smoother transition from old textbooks into the new Silver Burdett Series, Modern Arithmetic Through Discovery. Each concept presented is based upon familiar concepts, developed gradually, and directed toward particular fundamental principles.

The first concept developed is that of set theory and one-to-one pairing. A time allotment maximum of 1 week is given. The second concept developed is that of number. Included is: number-numeral distinction, order relations, place-values (decimal system) expanded notation, renaming numbers, number sentences, grouping by sixes, reading and writing base of six numerals, and counting in base six. The third is concepts in operations and includes: addition, subtraction, communtative principles, and associative principles for addition and multiplication and division. Fractions and decimals represent the fourth division with concepts of rising equivalent fractions and the decimal concept of hundredths. (The fifth and final unit discusses concepts of geometry, point, line segment, lines, plane, ray and angle.)

A glossary of modern mathematic terms is appended.

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Sets, Numbers and Operations

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AN INTERIM UNIT
IN CONTEMPORARY MATHEMATICS
FOR GRADE 6

Sets, Numbers, and Operations

Prepared
by

George R. Vojtko

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TO THE TEACHER

The teaching of arithmetic is undergoing many changes due to the impact of contemporary mathematics programs which have been conducted nationally during the past several years. It is anticipated that this trend will continue for some time in the future.

Developing an arithmetic curriculum during this period of rapid change poses many problems: new terms are being introduced, techniques of teaching are being modified, and arithmetic texts are continually being revised and updated. Many of the contemporary mathematics concepts are not included in the San Diego City Schools basic arithmetic text; however, they will appear in the next state textbook adoption which will be made available in September, 1965.

It is neither possible nor desirable to write directions and suggestions which will fit every teaching situation. Therefore, teachers should treat the content of this interim unit merely as a suggested guide to be implemented by their own resourcefulness.

Adjust the teaching procedure and activities described in this publication to your pupils and to your proven teaching techniques. Although teaching contemporary mathematics may be new to some of you, teaching children is your specialty.

Wm. H. Stegeman

William H. Stegeman
Assistant Superintendent
Curriculum Services Division

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PURPOSE

The purpose of this interim unit is to assist teachers and pupils in making the transition to a contemporary mathematics program and to the state textbooks that will be available in the fall of 1965. Experience has shown that any introductory work during the year preceding the introduction of a "new" program has made the first year considerably easier and more profitable for both teachers and pupils. Much of the content in this unit closely parallels the content of the textbook series,"Modern Arithmetic Through Discovery,"Silver Burdett Company.

Each concept presented in this unit is (a) based on familiar concepts, (b) developed gradually, and (c) directed toward certain fundamental principles. Pupils will not master these concepts immediately. However, experience and practice during this interim period will promote better understanding of the relationship emphasized in the new program.

SUGGESTIONS FOR USING THE UNIT

There are two ways in which this unit may be used:

1. To plan to complete the current state textbook (Winston) in the spring and teach the concepts in this unit in the remaining weeks of the year.

> More Able Group. Upon completing the work in the present state textbook (Winston), begin teaching the concepts suggested in this unit. It is anticipated that many teachers of grade six will start during the latter part of March. The teaching time should take from 7 to 10 weeks.
>
> Less Able Group. Whenever this group completes the work in the present state textbook, introduce them to this unit. However, in some instances, it will be necessary to omit some of the content in the Winston with this group so that the minimum time allotment (See page 3) may be devoted to a study of this unit.

2. To relate the teaching of concepts in this unit with the current state textbook during the remainder of the year.

In either case, it will be the teacher's responsibility to (a) become acquainted with the entire unit before any teaching is attempted, (b) spend a minimum amount of time on Chapters I, II, and IV and devote the major effort to those concepts in Chapters III and V, and (c) paraphrase much of the TEACHING PROCEDURE in the exploration phase, making modifications where necessary.

ORGANIZATION OF THE UNIT

Objective. The objective for teaching each concept is stated and includes the goal in understanding which the teacher should expect to achieve.

Background Information for the Teacher. The information under this heading is briefly stated and serves as a source of information which will be helpful to the teacher.

Teaching Procedures. This category is divided into two phases:

1. Exploration. This is largely an oral phase of the unit in which the teacher is provided with (a) questions which may be asked to promote inquiry and discussion, (b) examples which may be used to illustrate an idea, and (c) suggestions for use of concepts or semiconcrete materials. This is done prior to the time pupils are assigned any written exercises and is especially important with the less able groups.

2. Pupil Exercises. This phase of the unit is devoted to written exercises which are primarily aimed at reinforcing those skills and principles learned in the oral phase and practice phase of the lesson. After a concept is presented and developed, each pupil is provided with a worksheet. The same worksheet, with answers, is included in this guide. With less able pupils, the written work may be done in connection with oral reading, discussion, and use of illustrative, concrete materials.

DISCOVERY APPROACH

Children should be given an opportunity to "discover" many of the principles* of arithmetic which they will be expected to learn and use. Ideas which pupils discover for themselves, individually or through class discussion, make sense to them; rules which are committed to memory by rote methods are soon forgotten.

The discovery approach in teaching arithmetic can be fostered by the teacher who:

(a) Asks thought-provoking questions.

(b) Provides opportunities for discovery of the principles and structure of arithmetic through exploration with concrete materials.

(c) Encourages the search for patterns in a variety of number and geometric situations.

(d) Encourages studious inquiry through discussion of various ways of thinking about mathematical situations.

---

*See GLOSSARY, page 179, for a brief discussion of the mathematical terms used in this unit.

(e) Provides opportunities for testing of hypotheses through application of a theory.

(f) Withholds formal verbalization of the basic principles until they are understood operationally and used intuitively, allowing pupils to state ideas informally in their own language.

(g) Assists the pupil in making generalizations as a result of investigation.

When pupils learn arithmetic through their own discoveries, they develop (a) habits of logical thinking, (b) insight into social-mathematical situations, (c) accuracy and efficiency, and (d) ability to reconstruct an operation in arithmetic without relying wholly upon memory.

Results of recent research indicate that learning is improved when the generalization of a concept is discovered by the pupil rather than "handed" to him before he has had an opportunity to understand and use the skill related to the concept. This means that formal statements of generalizations should not be introduced prematurely but withheld until the concept has been developed and understood at an intuitive level.

SUGGESTED WEEKLY TIME ALLOTMENTS

Chapter	Weeks	
	Minimum	Maximum
I. Sets (pages 4-15)	$\frac{1}{2}$	1
II. Numbers (pages 16-63)	$1\frac{1}{2}$	2
III. Operations (pages 64-128)	3	4
IV. Fractions and Decimals (129-159)	1	$1\frac{1}{2}$
V. Geometry (pages 160-178)	1	$1\frac{1}{2}$
Total Number of Weeks	7	10

CHAPTER I

SETS

Many mathematical concepts were started thousands of years ago. Some of them probably were an outgrowth of man's need for a system which would help him keep account of such possessions as a flock of sheep. Others may have developed from his need to determine the size of objects and to find his position in relation to his surroundings.

When primitive man made marks or impressions on the ground and paired a mark on the ground with each of his animals, he developed a useful accounting system. This system helped him determine whether he had the same number of animals he started with, whether he had more animals, or whether he had fewer. Later, man developed a more efficient system of keeping an account of possessions; he designed a set of symbols which enabled him to name the number of them. Over the centuries other ideas that make communication and work easier were added to the "storehouse" of mathematical concepts. Mathematicians studied and refined the ideas, developing an abstract structure of interrelated concepts that is applicable in widely diverse situations.

Among the relatively new and very useful concepts in mathematics are those included in the study of SETS. The development of a formal theory of SETS began in the late years of the nineteenth century by Georg Cantor, a German mathematician. Concept of SETS were used to solve such difficult problems as: (1) "How many whole numbers are there?" (2) "How many points are there on a circle?" (3) "How many numbers are there between 1 and 2?" Solving equations, drawing graphs, describing geometric figures, etc., becomes simpler by using concepts and language of SETS.

Because the language of sets is an effective way to express ideas not only in mathematics but also in many other fields of knowledge, the study of sets is becoming increasingly important. The formal theory of sets is abstract and not started at the elementary school level. However, some of the simplest ideas and language are introduced to clarify concepts and facilitate oral and written communication.

In this unit, an elementary approach to the concept of sets is presented in terms of selected oral language and written symbolism. The word set is used to mean a "definite collection of things, objects, or ideas." When we speak of a set, we describe it precisely so that we can tell whether or not any given object or idea belongs to the specified set. The objects or ideas contained in a given set are called elements or members of that set.

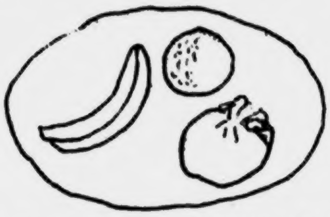
While there are many mathematical reasons for beginning the study of mathematics with the Concept of Sets, another powerful motivation for starting with sets is that "sets of objects" can be experienced with the senses and they can be easily manipulated. They afford the immediate concreteness that elementary pupils need.

A. Concept of Sets

Objective. To become better acquainted with the idea of a set and to begin building a "set language."

Background Information for the Teacher. To help us communicate our ideas about sets, we can describe sets in a number of ways:

1. Sometimes we draw pictures, using an outline to show that we are considering the set whose members are the objects pictured.



Such pictures help us to communicate both the idea of a set and to formulate a description of its members. Picturing sets is used to a considerable extent in working with children.

2. We can describe sets in words, for example, the set whose members are an orange, an apple, and a banana; our class; odd numbers from one through nine.

3. Frequently, we list or tabulate the elements of a set within braces. For instance, the symbol {1, 2, 3, 4, 5} is simply a name for the set described by the phrase "the set of consecutive counting numbers from one through five." {1, 3, 5, 7, 9} is a listing or tabulation of the set of odd numbers from one through nine.

We can describe the set which contains no elements. Consider the set which contains all United States senators under sixteen years of age. Clearly, there are no members in this set which will match the description. The set which has no elements is called the empty set. Using the brace notation, the empty set is represented by the symbol { }.

It is sometimes convenient to use a capital letter as a temporary name for a set. The statement A = {1, 2, 3, 4, 5} is read "A is the set of consecutive numbers from one through five."

It is important that we be able to translate a description into a set tabulation, but we must also be able to change a tabulation into a description. For example, the tabulation {a, e, i, o, u} can be described as (1) "the set of vowels in the English alphabet" or (2) "the set whose members are a, e, i, o, and u."

TEACHING PROCEDURE

Exploration. The classroom is an ideal setting for opportunities to talk about sets. Ask the pupils to describe different sets which are clearly visible to all pupils. Some of the responses might be:

(1) The set of chairs in the front row in our room.

(2) The set of maps at the front of our room.

(3) The set of books on the top shelf of our bookcase.

Refer to an object that does or does not belong to each set named and ask if it is a member of the set. For example, "Does my desk belong to the set of chairs in the front row? How do you know?"

On the chalkboard, write the numerals 2, 4, 6, 8. Ask the pupils to describe this set. Some of the responses might be:

(1) The even numbers two, four, six, eight.

(2) The set of even numbers beginning with two and ending with eight.

(3) The set of even numbers between one and nine.

Next, discuss with your pupils that a convenient way of referring to a set is to use braces. In so doing, you will be tabulating or listing a set. For example, show the following tabulated sets on the chalkboard:

$M = \{1, 2, 3, 4, 5\}$

$R = \{a, b, c\}$

$S = \{\text{Tuesday, Thursday}\}$

*$N = \{\qquad\qquad\}$

Now, ask the pupils to describe the sets M, R, S, and N shown above. Some of the responses might be:

M - The set of counting numbers less than six. The counting numbers beginning with one and ending with five.

R - The set whose members are the first three letters of the alphabet.

S - The set of days of the week beginning with the letter T, or the set whose members are Tuesday and Thursday.

N - The set consisting of "me."
The set whose member is "-----."

To develop the idea of the empty set, write the symbol { } on the chalkboard. Ask the pupils to describe this set. (Note: the idea of a set having no members may serve as a basis for discussion. An empty

---

*Write your name within the braces.

wallet, an empty specimen jar, or perhaps an empty pocket may be used to develop insight into such a set.) Some of the responses might be:

(1) The set without any members

(2) The set which is empty.

(3) The empty set.

Next, ask the pupils to give examples or to describe the empty set. Some examples are:

(1) The set of boys three years of age in grade 3.

(2) The set of girls nine feet tall.

(3) The set of even numbers between 10 and 12.

SUMMARY OF BASIC IDEAS

1. A set is a definite collection of things, objects, or ideas.
2. Each object or idea in a given set is a member or element of the set.
3. Pictures are sometimes used to illustrate a set.
4. Words are used (without braces) when describing a set.
5. Braces are used when tabulating a set.
6. A set may have a limited number of members, an unlimited number of members, or have no members; that is, it may be empty.

Pupil Exercises. Provide each pupil with worksheet No. 1.

The teacher may wish to do some exercises orally. Then have the pupils finish the exercises independently. Explain the instructions for each set of examples. Also, direct their attention to the use of the asterisk and footnote. Remind the pupils that braces are required when the directions call for listing or tabulating the members of each set. Some pupils find it difficult to make braces; therefore, frequent practice may be required.

Example 14 may be used as an optional item. The solution is: The two boys row across; boy A remains on shore while boy B returns for his father. The father rows across and sends boy A back to get boy B. Then both boys row across together.

Interim Mathematics Unit - Grade 6 Worksheet No. 1
San Diego City Schools (1-65)

LIST THE MEMBERS OF EACH SET. (Use braces when listing members of a set.)

1. The first seven letters of the English alphabet. {a, b, c, d, e, f, g}

2. The numbers counting by 4's beginning with 4 and ending with 28.
{4, 8, 12, 16, 20, 24, 28}

3. The Roman numeral for 5. {V}

4. The counting numbers greater than 4 and less than 12. {5, 6, 7, 8, 9, 10, 11}

5. The days of the week whose names begin with the letter T.
{Tuesday, Thursday}

USE THE WORDS FROM THE BOX BELOW TO LIST THE FOLLOWING SETS:

Mercury, Venus, Neptune, Pluto
astronaut, rocket, red, blue, Johnson

6. The two planets farthest from the planet Earth. {Neptune, Pluto}

7. The two planets closest to the sun. {Mercury, Venus}

8. The words that name a color. {red, blue}

9. The president of the United States. {Johnson}

10. The planet whose name begins with the letter X. { }

DESCRIBE THE FOLLOWING SETS:*

11. {January, June, July} The months of the year whose names begin with the letter J.

12. { } The empty set

13. {ball, bat, glove} Things used in playing the game of softball.

JUST FOR FUN

14. A man and his two sons wanted to cross a river in a small boat. The boat was too small to carry them all but would hold either the man or the two boys. All three managed to get across the river in the boat. Can you explain how they were able to do this?

---

*Descriptions will vary.

B. One-to-One Pairing

Objectives. (1) To provide a useful method for determining the relationship between sets. (2) To determine if given sets are equal, not equal, or equivalent.

Background Information for the Teacher. The idea of pairing objects is not new to any teacher. Moreover, it is quite likely that many of your pupils also have had informal experiences with pairing objects. For example, when a pupil is asked to distribute a sheet of paper to each member of the class, he is aware of a relationship between the paper and the pupil. When he matches a sheet of paper with each pupil in the class he has established a one-to-one correspondence; each pupil in the classroom has a sheet of paper and for each sheet of paper that was distributed there is a pupil.

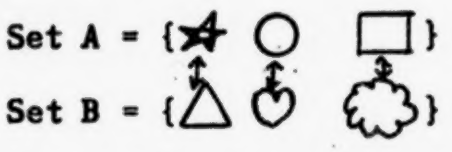
When a one-to-one correspondence exists between two sets, the sets are said to be equivalent. Centuries ago when shepherds kept account of their sheep by pairing a pebble with each member of a flock they were using the concept of equivalent and non-equivalent sets. If the sets were matching sets, that is, if there was a one-to-one correspondence between pebbles and sheep, they knew that all the sheep had returned from pasture. When there were pebbles "left over" they knew that some sheep were missing; fewer sheep were in the flock than originally. When there were sheep "left over" after the pairing process was completed, they knew there were more sheep in the flock than at a prior time.

A comparison of any two sets can be made by using the pairing process. Thus, one can determine, without counting, which of the sets has more members, which has fewer members, or whether the sets are equivalent.

Note that when a one-to-one correspondence exists between two different sets, the sets are equivalent, not equal. Equal sets are sets that have exactly the same members. For example, suppose we designate the set of children who serve on the clean-up committee as Set C. Lynn, Bob, and Sara are the members of the clean-up committee. We can say that the set of children on the clean-up committee equals the set of children consisting of Lynn, Bob, and Sara. Or, since we named this set C, we can write: C = {Sara, Bob, and Lynn}. We are using different names to specify the identical set. The order of listing the members is irrelevant.

It is so obvious that equal sets are also equivalent that discussion of the idea sometimes seems confusing. The following illustrations may serve to clarify it.

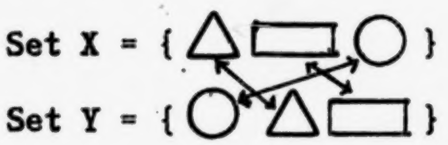
1. Can Set A be placed in one-to-one correspondence with Set B?
   Is Set A equal to Set B?



The members of Set A and Set B can be paired and we "come out even." Since the sets are in one-to-one correspondence, they are equivalent.

The members of Set A are a star, a circle, and a square. The members of Set B are a triangle, a heart, and a flower. The sets are not the same, therefore they are not equal.

2. Can Set X be placed in one-to-one correspondence with Set Y?
   Is Set X equal to Set Y?



The members of Set X and Set Y can be paired and we "come out even." Since the sets are in one-to-one correspondence, they are equivalent.

The members of Set X are a triangle, a rectangle and a circle. We find pictures of the same triangle, rectangle, and circle in Set Y. The sets are identical, therefore they are equal.

The distinction between equivalence and equality of sets has been pursued at some length because it is basic to the concept of equality that is developed throughout the mathematics program. The equal sign (=) in mathematics is the mathematical symbol for "is identical with." When we consider the number property of the equivalent sets in the examples above, we see that the number of elements in each is, in fact, identical.

We could use the word three, the Arabic numeral 3, the Roman numeral III, and other names for the number of each set, and correctly say that three = 3, etc., because we are naming the same number. Since the concepts of equality and inequality are fundamental in mathematics, it is important for the teacher to use the words equivalent and equal correctly in reference to sets of objects.

TEACHING PROCEDURE

Exploration. (One-to-One Pairing) Ask your pupils to pretend that they know nothing about counting and that they are without a written language. Solve the following problems: *

1. "Show us how you can tell, without counting, whether or not the set of fingers on one hand is equivalent to the set of fingers on the other hand."

---

*Answers will vary.

They are not the same fingers, but they can be matched one-to-one.

2. When we all leave the room for relief or recess, how can we tell that upon returning, all pupils can be accounted for?"

   Before leaving, check to see what chairs are vacant. Upon returning, if the same chairs are vacant, we can assume that all have returned.

   (Discuss the fact that we are assuming only that no one has entered the room while we were out.)

3. Suppose you are interested in determining whether there are more girls than boys in your classroom. What would you do?"

   Ask the girls to line up on one side of the classroom and the boys to line up on the opposite side. Then match or pair each girl with a boy. If the sets match one-to-one, that is, are equivalent sets, the number will be the same; if they do not match, there may be more boys than girls or more girls than boys.

4. Provide each pupil with a token (strip of paper, bottle cap, pebble, marble, or other small object. When the pupils leave the room for the playground one by one, have each pupil deposit a token into a container which has been placed by the door. Upon returning, have each pupil pick up one token. With the activity carried out (providing all pupils have returned), ask the following: "How does this compare with the way primitive man kept account of his sheep? With the distribution and collection of report cards?"

   The method is the same; the examples are similar.

Exploration. (Equal sets). On the chalkboard write a verbal description such as "the set of national holidays during the month of February." Ask the pupils, "Name the members of the set." The responses might be:

{ Lincoln's Birthday, Washington's Birthday }

Next, point out that although it is possible to use words, pictures, or braces to describe sets (using commas between members to eliminate confusion between two or more elements), capital letters are used to name sets. Refer to the set {Lincoln's Birthday, Washington's Birthday}; label the set as A. The set should now be represented as:

A = {Lincoln's Birthday, Washington's Birthday}

Make it clear that the choice of letter name is an arbitrary matter in that A could just as well have been Q. Explain that a verbal description of A would be: "A is the set whose members are Lincoln's Birthday and Washington's Birthday," or "A is another name for the set whose members are Lincoln's Birthday and Washington's Birthday."

On the chalkboard show that a set could be tabulated (listed) in other ways. For example:

(1) A = {Lincoln's Birthday, Washington's Birthday}

(2) A = {Washington's Birthday, Lincoln's Birthday}

Lead the children to conclude that each of these tabulations names the same set. Ask the pupils to describe A, example (1) and example (2) above. Responses might be:

(1) A is the set whose members are Lincoln's Birthday and Washington's Birthday.

(2) A is the set whose members are Washington's Birthday and Lincoln's Birthday.

Next, ask, "Do examples (1) and (2) describe the same set? Yes What can we say about the order of listing the members of each of the sets?"

The order of listing the members in the set does not change the description.

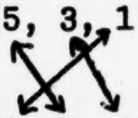
To develop the idea of "equality," write the following example on the chalkboard:

M = {5, 1, 3}

N = {1, 3, 5}

Explore with your pupils using (a) the recent discovery that the order of listing the members does not change the set described and (b) the concept of one-to-one correspondence. Use the example already on the chalkboard; draw the lines, matching names for the same number.

M = {5, 3, 1}



N = {1, 5, 3}

Indicate to your pupils that when we write M = N, we mean that M and N are different names for the very same set.

Have the pupils consider:

A = {1, 2, 3, 4, 5}

B = {5, 4, 3, 2, 1}

Ask, "Why is it correct to say that A = B?" Some of the responses might be:

(1) A has the same members as B.

(2) The set described by A is the same set as that described by B.

(3) Each member of A can be matched with the same number in B.

Exploration. (Equivalent Sets). The notion of a pair of sets being in one-to-one correspondence leads to the idea of equivalence. Two sets which can be placed in one-to-one correspondence with each other are said to be equivalent. For example:

S = {□ △ ☆}

T = {⊠ ◇ ▷}

Ask, "What general statements can we say about Set S and Set T?" (Ask the pupils to keep in mind such ideas as (1) description of a set, (2) one-to-one correspondence (matching), and (3) equal sets.)

(1) S is the set whose members represent a square, a triangle, and a star.

(2) The description of S does not match the description of T.

(3) S and T can be matched one-to-one.

(4) S and T are not the same in all respects.

(5) Sets S and T are not equal.

Indicate to the pupils that when two sets can be matched one-to-one, they are said to be equivalent, whether or not the members are the same or different.

To develop the idea of equivalent sets using other examples, write the following on the chalkboard:

X = {a, b, c}

Y = {1, 2, 3}

Z = {□ ☆ ○ △}

Ask, "In comparing sets X, Y, Z (only two sets at a time may be compared) what can we say about them?" Some of the responses might be:

(1) X is equivalent to Y; Y is equivalent to X.

(2) Y is not equivalent to Z; X is not equivalent to Z.

(3) X is not equal to Y; Y is not equal to Z.

(4) X is not equal to Z; Z is not equal to X.

SUMMARY OF BASIC IDEAS

1. A comparison of any two sets may be made by using the process of pairing one-to-one.

2. Sets are in one-to-one correspondence when the members of one set can be matched, one-to-one, with the members of another set.

3. The word "equal," when referring to sets, is intended to mean "the same set as."

4. Sets may be assigned a capital letter name. For example, when we write A = B, we mean that _A_ and _B_ are different names for the very same set.

5. The order of the names or members of a given set is irrelevant when comparing two sets.

6. Sets are equal whenever each member of one set is also a member of the other set, in one-to-one correspondence irrespective of order, and there are no other members involved.

7. Sets are equivalent whenever each member of one set can be matched with the members of another set, the members being same or different.

8. If two sets are equal, they are also equivalent. However, if two sets are equivalent, they are not necessarily equal.

Pupil Exercises. Provide each pupil with worksheet No. 2.

Discuss the worksheet with the pupils before requiring independent work. In examples 5 and 14, the order of the members may vary. Explain example 13 and the use of the asterisk.

COMPARE THE FOLLOWING SETS:

M = {zero, one, two}
P = {two, one, zero}
N = {four, six, eight, ten}
Q = {5, 6, 7, 8}

1. Which set is equal to M? P
2. Which sets are equivalent but not equal? N and Q
3. Zero is a member of set M and set P.
4. Which sets are both equivalent and equal? M and P
5. List two sets which would be equivalent to Q. {a, b, c, d}
{1, 2, 3, 4}

USE THESE SETS TO ANSWER THE FOLLOWING QUESTIONS. WRITE "true," "false," OR "none" IN EACH BLANK SPACE.

X = {5, 10, 15, 20, 25}   Y = {3, 6, 9, 12, 15}   Z = {0, 10, 20, 30}

6. X is equivalent to Y. true
7. Y and X can be matched one-to-one. true
8. X is not equal to Y. true
9. Which sets are equal? none
10. 15 is a member common to X, Y, and Z. false
11. 20 is a member common to X and Z. true
12. 0 is a member of Z but not X. true

COMPLETE EACH QUESTION BY LISTING THE SET. *

13. Set B is equal to {r, s, t, u, v}. Think of a set that has the same number of members as B but is not equal to B. Call it set R.

R = { a, b, c, d, e }

14. Set N is equal to {1, 4, 9, 16, 25, 36}. Think of a set that is equal to N. Call it set M.

M = { 4, 9, 16, 25, 36, 1 }

---

*Answers will vary.

CHAPTER II

NUMBERS

A. Concept of Number

Objective. To emphasize that a number is an idea which is associated with sets of objects.

Background Information for the Teacher. If we stop to think about the process of counting, we quickly realize that it is nothing more than establishing a one-to-one correspondence between an ordered set of names of the counting numbers and the set we are counting. (Counting numbers are: 1, 2, 3, 4, etc., that is, they begin with 1 and continue indefinitely. They are sometimes called natural numbers.) The last number named in the set of counting numbers gives us the number of the set we are counting. This process is an improvement over the simple one-to-one matching of sets. It tells us not only that the sets have the same number but also what the number is.

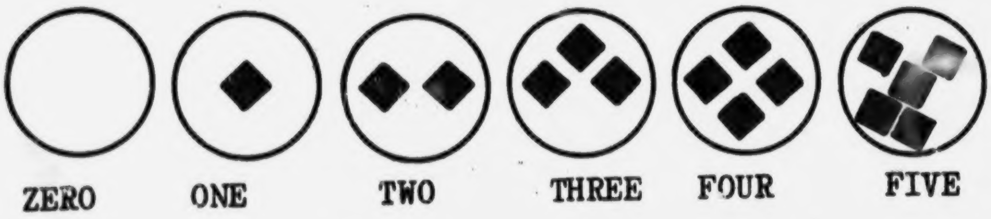
Suppose a pupil in your classroom is counting a set of objects such as a 12" ruler, compass, protractor, and a flannel board. To determine the "number" property or "how many" objects he has, he may begin by mentally matching the objects with the set of counting numbers. Thus,

Objects:	ruler	compass	protractor	flannel board
	↕	↕	↕	↕
Names of Counting Numbers:	1	2	3	4

After completing this matching, he can conclude that "there are 4 objects in this set."

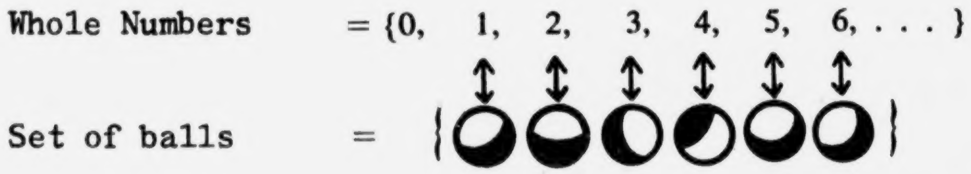
TEACHING PROCEDURE

Exploration. On the chalkboard represent a convenient model together with the number name such as the following illustration:

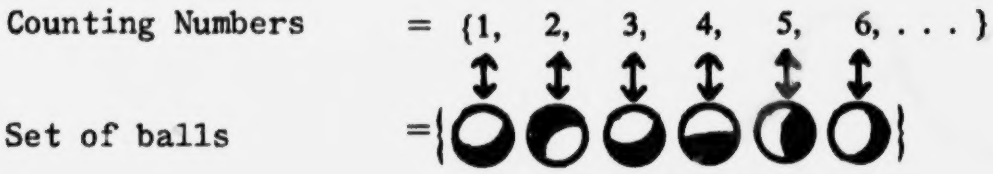


The pattern which develops shows clearly that the orderly sequence begins with the empty set and increases by "one more" member in each succeeding set. Clearly, this process can be continued indefinitely. Once an orderly system has been created and names and numbers assigned to these sets, we have the machinery with which to count.

To clarify the process of determining the number of members in a given set, inform the pupils that a standard set has to be established. (Whole numbers are sometimes called cardinal numbers.) The whole numbers are used in the following example:



Have the pupils notice that whenever we are counting a set containing one or more members, the first member is matched with the whole number one, the next with number two, etc.; an empty set would be associated with the whole number zero. Thus, the whole number (shortened to "number") of the set of balls is represented by the symbol 6. Illustrated in terms of one-to-one correspondence with the counting numbers we have the following:



Thus, the last number in the set of counting numbers matched with the set of balls indentifies the number of the set as represented by the symbol 6. Therefore, the standard set which determines the number of members in a set is the set of whole numbers. In counting "one and one more," the set of counting numbers is used.

As a group activity, to further develop the number idea ("how many" members are in a set), have the class compare the number of the set of red stripes in the United States flag with the number of the set of white stripes. 7 red and 6 white

Another activity* might be to cut out fifty-five (55) felt objects of one uniform size and place them on the flannel board. (Teachers may prefer to use suitable representative markings on the chalkboard such as circles ○ , squares □ , or triangles △). Have several pupils separate the fifty-five objects into ten sets whose numbers are 8, 7, 5, 10, 6, 9, 4, 1, 3, and 2. Next have the pupils arrange the sets in order, so that each set contains one more object or member than the set before it. Thus, the orderly arrangement should look like this:

---

* This activity may be modified or replaced by another suitable example. The purpose of this activity is to show the orderly arrangement of the development of the number sequence and the relationship of the empty set.

0	1	2	3	4	5	6	7	8	9	10
	x	x	x	x	x	x	x	x	x	x
		x	x	x	x	x	x	x	x	x
			x	x	x	x	x	x	x	x
				x	x	x	x	x	x	x
					x	x	x	x	x	x
						x	x	x	x	x
							x	x	x	x
								x	x	x
									x	x
										x

Discuss where the empty set should be placed in an orderly arrangement of sets. Help pupils to see that the empty set is logically a part of the arrangement. It precedes the set with one member since the set with the number one has one more member than the empty set.

Point out to the class that so far they have been working with sets of objects and the set of whole numbers. Now they will be working with sets of numbers and not the objects. Ask, "Who would come to the chalkboard and tabulate the whole numbers between 3 and 6?" { 4, 5 } Next, ask, "Who could tabulate the first three whole numbers in order?" { 0, 1, 2 } Now, raise the question, "Is it possible to tabulate all of the whole numbers?"

Have a pupil come to the chalkboard and make a long list of the whole numbers beginning with 0. Lead the class to discover that no matter where the list stops, there will always be another number greater than the last one listed or tabulated. Explain that a list of all the numbers would be endless. Now refer back to the question that was raised, "Is it possible to tabulate all the whole numbers?"

No, because there are more numbers than can be counted.

Show the class a way to indicate that the list of whole numbers goes on endlessly; that is, {0, 1, 2, 3, 4, ...}. Explain that the three dots placed at the end of a number pattern indicate that the numbers go on without end. (With fifth-grade pupils, avoid using the words "infinity" and "infinite.") Emphasize that three dots may indicate missing members if they appear in the midst of a pattern as shown in this example: 2, 4, 6, ..., 12, 14, 16.

Next, ask someone to count aloud the number of boys in the room. Ask, "Which of the whole numbers is not used in counting?" Zero Explain that the set of whole numbers beginning with one has a special name. It is called the set of counting numbers, that is, {1, 2, 3, 4, ...}.

SUMMARY OF BASIC IDEAS

1. "Number" is a property of every set; that is, a set may have no members, one member, two members, etc. Every set has a whole number associated with it.
2. The "number property" of a set is determined by matching the members of a set with a standard set.
3. The set of whole numbers is {0, 1, 2, 3, 4, ...}; that is, it begins with 0 and continues in succession indefinitely and consecutively.
4. The set of counting numbers is {1, 2, 3, 4, ...}; that is, it begins with 1 and continues indefinitely and consecutively.
5. A set may have a definite (finite) number of members or an indefinite (infinite) number of members.

Pupil Exercises. Provide each pupil with worksheet No. 3.

Before assigning this worksheet, make sure the pupils understand the instructions related to these exercises. Braces are not provided; therefore, pupils must write their own for examples 11 through 20. Providing a brief period for practicing braces before the assignment will prove helpful.

IN COLUMN A, THE MEMBERS OF THE SETS ARE LISTED. WRITE THE NUMERAL THAT TELLS "how many" MEMBERS ARE IN EACH SET. WRITE YOUR ANSWER IN THE BLANK SPACE. THE FIRST ONE IS DONE FOR YOU.

	Column A	Number of Members
1.	R = {△, ○, □, ⬡, ✡, ☽}	R = 6
2.	S = {u, v, w, x, y}	S = 5
3.	T = {0, 1, 2, 3, ..., 13}	T = 14
4.	U = { 0 }	U = 1
5.	V = { }	V = 0
6.	W = {1, 2, 3, 4, ..., 29}	W = 29
7.	X = {5, 10, 15, 20, ..., 50}	X = 10
8.	Y = {100, 101, 102, 103, 104}	Y = 5
9.	Z = {1, 2, 3, 4, ..., 999}	Z = 999
10.	M = {0, 1, 2, 3, 4, ..., (99,999)}	M = 100,000

LIST THE MEMBERS OF EACH OF THE FOLLOWING:

11. The set of whole numbers greater than 5 can be listed as {6, 7, 8, ...}. Three dots are used to indicate and so on. Now, use this method to list the members of the set of whole numbers greater than 70.
{71, 72, 73, ...}

12. List the set of even numbers between 0 and 2. { }

13. List the set of odd numbers beginning with 3 and ending with 39.
{3, 5, 7, 9, ..., 39}

14. List the set of whole numbers less than 5. {0, 1, 2, 3, 4}

15. What is the first counting number? {1}

16. What is the first whole number? {0}

17. List a set of even numbers between 17 and 23. {18, 20, 22}

LIST THE MEMBERS OF {2, 4, 6, 8, ..., 40} WHICH CAN BE:*

18. Divided by 5. {10, 20, 30, 40}

19. Divided by 2. {2, 4, 6, ..., 40}

20. Divided by 3. {6, 12, 18, 24, 30, 36}

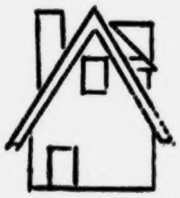
---

*Without remainders.

B. Number-Numeral Distinction

Objective. To develop an understanding of the use of the term "number" which represents the numerical concept of quantity, and the term "numeral" which represents the symbol used to name the number.

Background Information for the Teacher. Children learn number names in connection with the number of objects in a set long before they can read or write the standard symbol or numeral. After they learn to write the numerals, they must realize that these symbols are names for numbers; they are not the numbers themselves. One way of expressing this is to say that numerals name or represent numbers. We interpret this in the same sense that each of the two visible symbols,



and house

represent a house. Most certainly, neither symbol is a "real" house.

> Number is an idea, an abstraction which arises from situations involving sets of objects.
>
> Numeral is a symbol by which we refer to a number.
>
> There are many ways of naming the same number, that is, there are many numerals for each number.

Perhaps the simplest way to explain the idea is to say, "Numerals are names for numbers." Write the word "Bob" on the chalkboard and ask the pupils what you have written. If some of them say "a boy," invite them to come to the chalkboard and play with "Bob." They will easily recognize that "Bob" is not a boy, but merely the name for a boy. Similarly, 2 is not a number but merely a visible name or symbol for a number.

If man lived alone in this world, the number idea would have been sufficient. But since man is a social being, he wishes to communicate his thoughts to other human beings. This is why numerals were invented.

Another point of interest which requires our attention and consideration is that a number idea or number may be expressed in many ways since a number has many names. For example, "Six," "VI," "4 + 2," "12 ÷ 2," and "3 × 2" are all names for the same number, since they are all numerals that name (or represent) the same number.

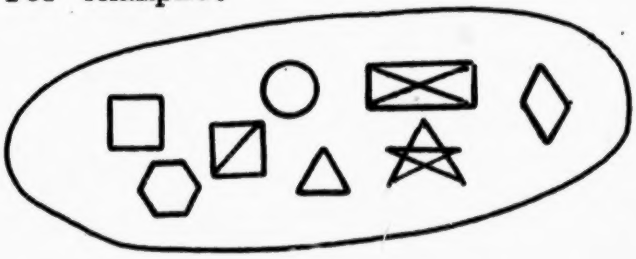
There may be some misgiving concerning the numerals "4 + 2" and "3 × 2." Yet, what number do you think of when you see "4 + 2" and "3 + 2"? In either case you think of the number we call "six." Hence, each of these communicates the idea of "six" and is therefore a numeral (or name) for the same number. The expression "4 + 2" expresses 6 as a sum and "3 × 2" expresses 6 as a product.

The study of arithmetic has too often meant merely the development of computational skills. Little consideration has been given to the nature of the "things" with which we compute. As an outgrowth of this emphasis, many feel an irresistable urge to "do an exercise" when they see an expression such as 4 + 2. This "urge to compute" or simplify compels some pupils to replace 4 + 2 by 6. However, 4 + 2 is a correct name for a number as is 6.

It is important for pupils to be able to write other names for 6. Sometimes the expanded names 4 + 2, 7 - 1, 24 + 4, etc. are more useful than the concise form; pupils need to be able to find a useful expanded form as well as the most concise form.

TEACHING PROCEDURE

Exploration. On the chalkboard, draw a picture to represent a set of eight objects, for example:

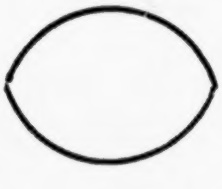


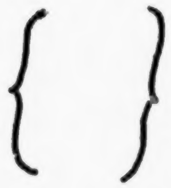
Ask pupils to come to the chalkboard and write different numerals to represent the number of the set. Some of the responses might be:

8, 7 + 1, 4 + 4, 4 × 2, 16 + 2, VIII

Lead the class to generalize that many numerals may be used to name the same number. Point out that the numeral 8 is the simplest numeral for the set illustrated above.

Continue the activity by asking for various numerals to name other sets pictured on the chalkboard. Include practice in writing numerals for the number of the empty set as shown below:





Some of the responses might be:

$4 \times 0, \quad 20 - 20, \quad 0 - 0, \quad 8\frac{1}{3} - 8\frac{1}{3}, \quad 0, \quad$ etc.

Inform the pupils that all of the written work at the board required symbols or numerals to represent the number idea.

Next, tell the pupils that you are thinking of the number "12." Ask two pupils to come to the chalkboard and have each pupil write several different numerals for the number "12." Responses for pupil A and pupil B might be:

Pupil A	Pupil B
$6 + 6$	3×4
$8 + 4$	$24 \div 2$
$10 + 2$	$12 \times 1 \times 1$
$14\frac{1}{2} - 2\frac{1}{2}$	$32\frac{1}{4} - 20\frac{1}{4}$

Now, write an equal sign (=) between each of the two corresponding numerals. For example:

$$6 + 6 = 3 \times 4$$
$$8 + 4 = 24 \div 2$$
$$10 + 2 = 12 \times 1 \times 1$$
$$14\frac{1}{2} - 2\frac{1}{2} = 32\frac{1}{4} - 20\frac{1}{4}$$

Help pupils realize that the equal sign indicates that two names are being used for the same number. Next, have several pairs of pupils name numbers in two different ways and use the equal sign to show that the two numerals they have chosen represent the same member. Encourage the use of a variety of numerals and operations. For example:

Number Idea	Numeral Used to Express the Number		
8	$16 - 8$	or	$4 + 4$
14	$5 + 9$	or	$4 + 10$
10	$15 - 5$	or	$16 - 6$
6	$12 \div 2$	or	2×3
25	$13\frac{1}{3} + 11\frac{2}{3}$	or	$12.50 + 12.50$

Point out to the class that the idea of number may be communicated by spoken words as well as by written symbols. Thus, the concept that a number is an idea and that a numeral is used to communicate the idea will be reinforced.

SUMMARY OF BASIC IDEAS

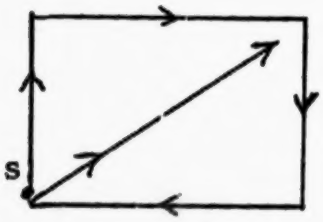
1. "Number" is an "idea"--an abstraction which arises from a situation involving sets of objects.
2. "Numeral" is a symbol which is used to communicate the number idea; numerals name numbers.
3. A number has many names which can be written in many ways.
4. An equal sign (=) is used to show that two different numerals name the same number.

Pupil Exercises. Provide each pupil with worksheet No. 4.

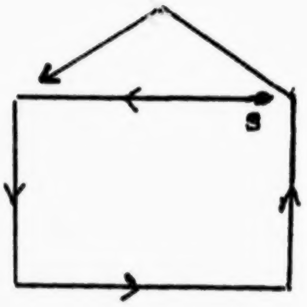
Have the pupils read the instructions carefully before attempting the exercises independently. With the less able group, it may be necessary to interpret the instructions by doing several of the examples on the chalkboard. Encourage the use of addition, subtraction, multiplication and division in examples 6 through 11.

The "Just for Fun" examples are optional. If time permits, they may be used as a basis for group discussion. Here are the possible approaches to (a), (b), and (c); (d) is impossible.

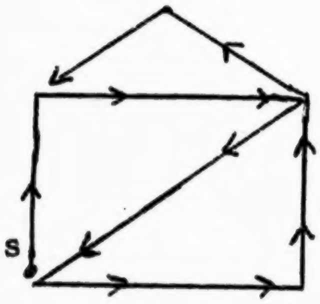
(a)



(b)



(c)



(d)

Impossible

Worksheet No. 4

"If the number sentence is true, an equal sign between two numerals means that they name the same number. For example:

$35 + 25 = 60$ $29 = 15 + 14$

The numeral 35 + 25 and the numeral 60 both name the number sixty. Likewise, the numeral 29 and the numeral 15 + 14 both name the number twenty-nine."

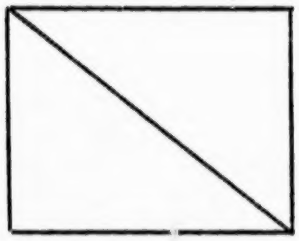
FOR EACH NUMERAL IN COLUMN M, CHOOSE A NUMERAL FROM COLUMN N. SHOW THAT THE CHOSEN NUMERALS NAME THE SAME NUMBER.

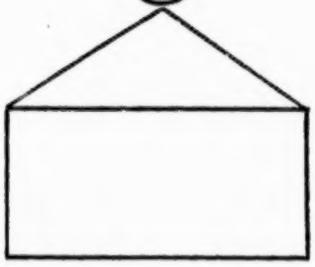
	Column M	Column N	Names for the same number
1.	45	25 + 25	45 = 20 + 25
2.	75 − 25	3 + 3 + 1	75 − 25 = 25 + 25
3.	$3\frac{1}{2} + 2\frac{1}{2}$	20 + 25	$3\frac{1}{2} + 2\frac{1}{2}$ = 10 − 4
4.	2.5 + 4.5	12 + 12	2.5 + 4.5 = 3 + 3 + 1
5.	6 × 4	10 − 4	6 × 4 = 12 + 12

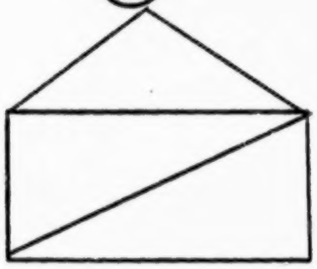
IN THE BLANK SPACES, WRITE THREE DIFFERENT NAMES FOR THE NUMBER NAMED AT THE LEFT. THE FIRST ONE IS DONE FOR YOU.*

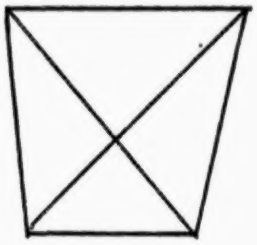
6.	36	30 + 6	18 × 2	72 ÷ 2
7.	$38\frac{1}{2}$	$38 + \frac{1}{2}$	$\frac{1}{2} + 38$	$36 + 2\frac{1}{2}$
8.	160	160 × 1	100 + 60	80 + 80
9.	375	300 + 75	400 − 25	350 + 25
10.	673	673 × 1	0 + 673	600 + 73
11.	979	1000 − 21	900 + 79	900 + 70 + 9

JUST FOR FUN. CIRCLE THE LETTER OF EACH DIAGRAM WHICH MAY BE DRAWN WITHOUT LIFTING YOUR PENCIL FROM THE PAPER AND WITHOUT TRACING A PREVIOUSLY DRAWN LINE. (Show your work.)


[✓ (a)]


[✓ (b)]


[✓ (c)]


(d)

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*Answers will vary.

C. Order Relations

Objective. To show that our system of expressing number ideas is orderly and is useful in comparing numbers.

Background Information for the Teacher. Suppose a pupil opens a box of plastic or felt cut-out numerals and scatters them on the table. If he knew nothing about our numeration system, he would probably put them in any order, that is, 3, 4, 5, 1, 2, 6, etc. However, having some experience with our numeration system, he probably would arrange them in the order 1, 2, 3, 4, 5, etc.

The concept of order is very important in mathematics just as it is in any field, be it scientific or non-scientific. For example, given a set, we can often arrange the elements in some "agreed upon" order.

1. Natural order of the counting numbers:

$$\begin{array}{cccccc} 1, & 2, & 3, & 4, & \ldots\ldots, & n \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow \\ \text{first} & \text{second} & \text{third} & \text{fourth} & \ldots, & n^{th} \end{array}$$

2. Order of the letters of the alphabet:

$$\begin{array}{cccccc} a, & b, & c, & d, & \ldots\ldots, & z \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow \\ \text{first} & \text{second} & \text{third} & \text{fourth} & \ldots & \text{twenty-sixth} \end{array}$$

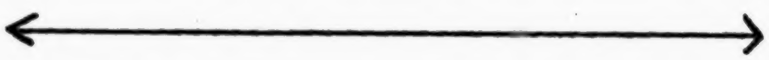
The set of whole numbers has a natural order. They can be so arranged that each number is one greater than the preceding number: 0, 1, 2, 3, 4, 5, ...

TEACHING PROCEDURE

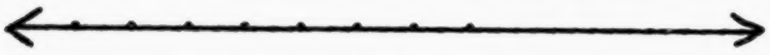
Exploration. Indicate to the pupils that one way to show the orderly arrangement of the set of whole numbers is the "number line."

Simply represent a line as illustrated (a) with arrows on both ends indicating that it can go on and on as far as we want to extend it.

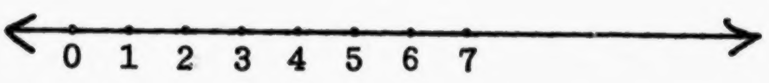
(a)



(b)

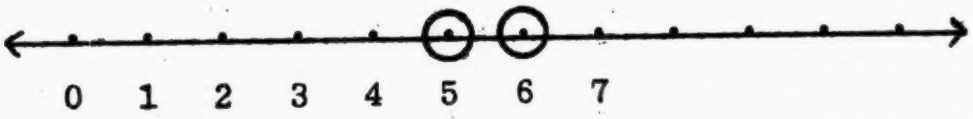


(c)

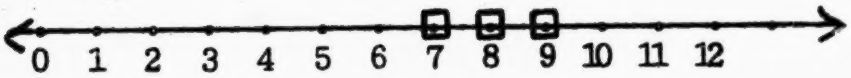


Then we mark on it, as in illustration (b), preceding page, a number of equally spaced marks or points. Starting at the left, we label the points with the whole numbers in order, that is, 0, 1, 2, 3, 4, 5, 6, 7, ... The arrows on the number line indicate we can extend it if we want to; the three dots ... at the end of our list of numbers indicate that we can extend it, also. Now the order of the whole numbers shows up clearly by the position of the labeled points. The point labeled 7 is to the right of the point labeled 5; 12 would be to the right of 11; 23 would be to the left of 27; etc. Thus we can see that as we move to the right, the numerical value increases; as we move to the left from a given point, the numerical value decreases. Because of this orderly arrangement, we can show that 27 is greater than 26; 29 is greater than 28; 35 is less than 36; 43 is less than 45; etc.

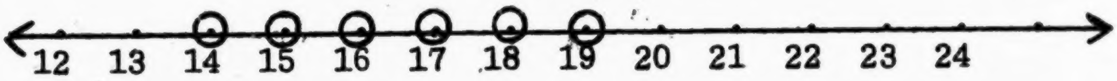
Now, ask a pupil to come to the chalkboard and draw circles around the points that represent the whole number(s) less than 7 and greater than 4.



Next, have another pupil draw a box or square-shaped frame around the points that correspond to the whole numbers which are greater than 6 and less than 10 (extend the number line for this if necessary).



Construct another number line, this time beginning with twelve; have pupils write numerals to name the points on this line. Have someone circle the points that represent the whole numbers greater than 13 and less than 20, etc.

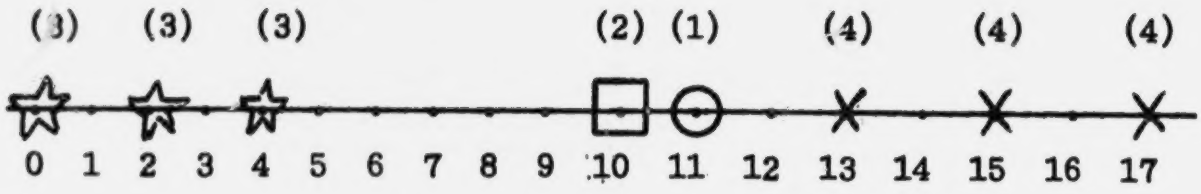


Vary this activity until pupils can readily work with situations similar to those mentioned above.

Now, construct a number line, showing equal unit segments, and show the whole numbers corresponding to the points on the line. Ask pupils to come to the chalkboard and locate the points which correspond to the numbers named by the following:

(1) Circle (●) the whole number that is greater than 7 + 3.
(2) Frame (■) the whole number that is 2 less than 24 + 2.
(3) Star (★) the even whole numbers that are less than 3 × 2.
    (0 is considered an even whole number.)
(4) (X) the odd whole numbers that are greater than 4 + 7.

Problems:



Proceed in this way to give pupils several opportunities to find other points on a number line.

SUMMARY OF BASIC IDEAS

1. The concept of order is important in mathematics just as it is in any field, be it scientific or non-scientific.
2. The set of whole numbers has a natural order.
3. The number line is one of many ways in which we can illustrate the orderly arrangement of whole numbers. It is useful in analyzing and solving many number situations.

Pupil Exercises. Provide each pupil with worksheet No. 5.

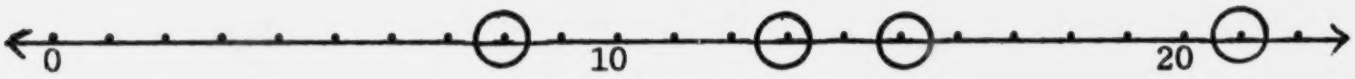
Discuss the instructions with the pupils before assigning this worksheet. Use chalkboard illustrations, especially with the less able group, to clarify the instructions.

Work examples 9 and 10 cooperatively with the pupils. Those who feel they can do the examples independently should be encouraged to do so.

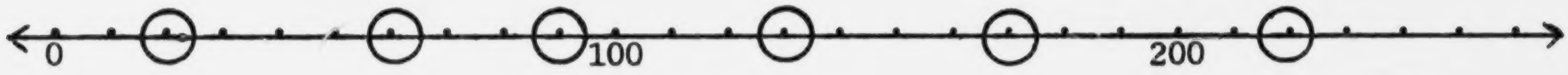
"Locating Points on the Number Line"

DRAW A CIRCLE AROUND THE POINT WHICH CORRESPONDS TO EACH MEMBER OF THE SET OF NUMBERS NAMED IN EACH EXAMPLE.

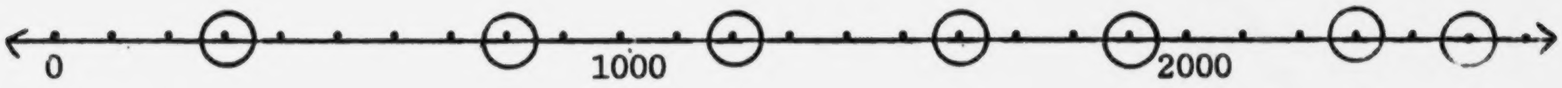
1. {8, 13, 15, 21}



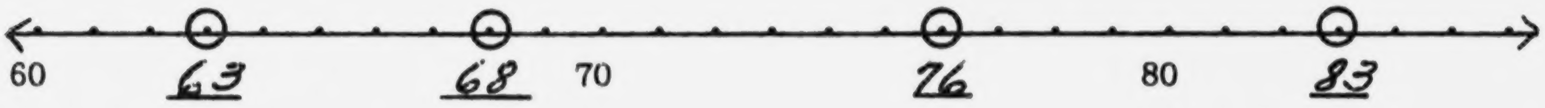
2. {20, 60, 90, 130, 170, 220}

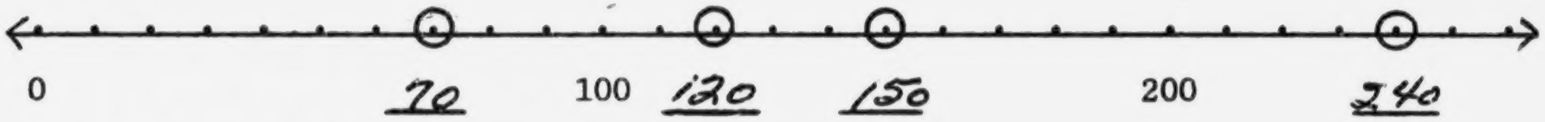


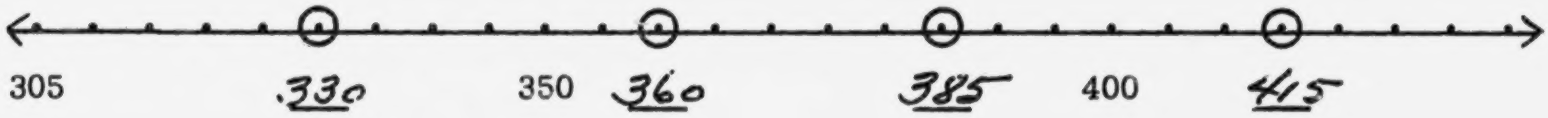
3. {300, 800, 1200, 1600, 1900, 2300, 2500}

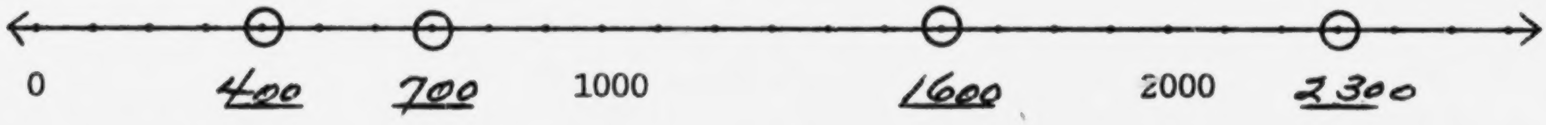


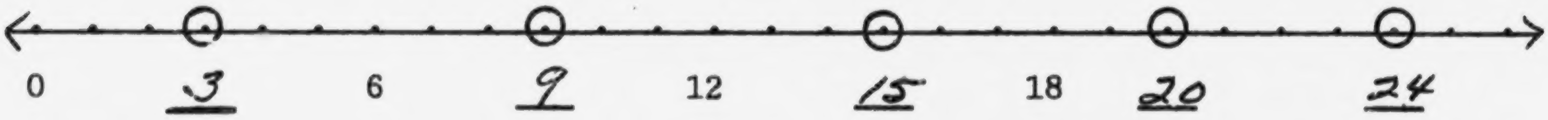
WRITE THE CORRECT NUMERAL BELOW EACH CORRESPONDING POINT WHICH IS CIRCLED.

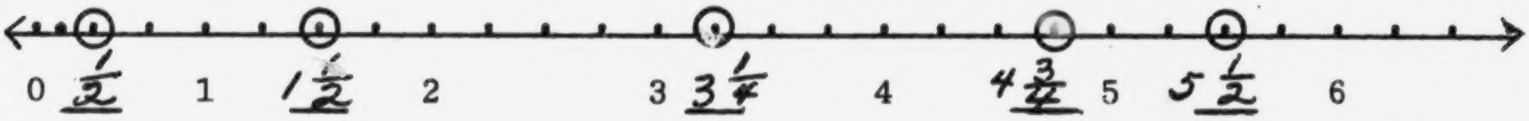
4.

60
63
68
70
76
80
83

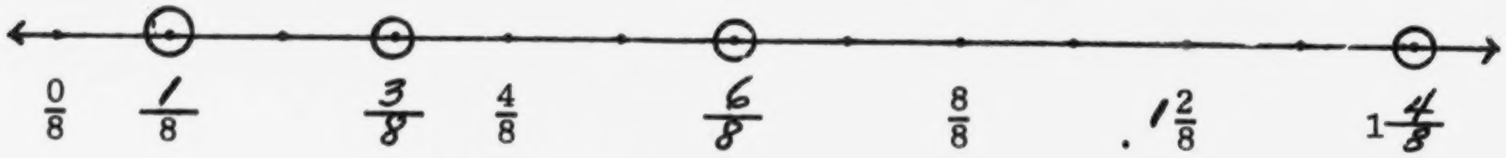
5.

0
70
100
120
150
200
240

6.

305
330
350
360
385
400
415

7.

0
400
700
1000
1600
2000
2300

8.

0
3
6
9
12
15
18
20
24

*9.

0
1/2
1
1 1/2
2
3
3 1/4
4
4 3/4
5
5 1/2
6

*10.

0/8
1/8
3/8
4/8
6/8
8/8
1 2/8
1 4/8

---

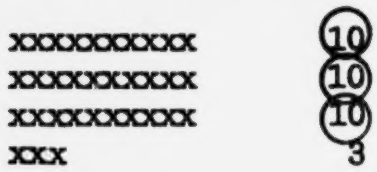
*Try this one even though you may not be familiar with fractions on the number line.

D. Place Value (Decimal System)

Objective. To show that our system of numeration, because of the inherent place-value property, makes it possible for us to name any whole number without using more than ten distinct symbols or digits.

Background Information for the Teacher. The place-value scheme that our system utilizes has a base of ten. Base ten means that we group by tens. That is, given a set of objects (see Figure 1) below:

Figure 1:



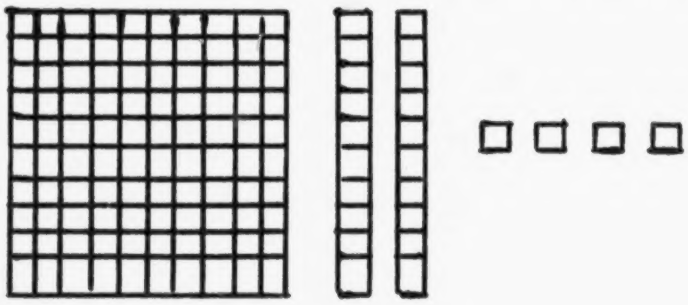
we might ask, "How many sets, each having ten members, can be formed?" We see that there are 3 sets of ten and 3 left over. The importance of place-value is evident when we attempt to write the numeral for the arrangement of objects (x's) in illustration A.

To name the number of any set of objects in our base-ten system of numeration, we use ten different basic symbols (sometimes called digits). These symbols are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. In decimal (base-ten) place-value numeration the number of objects in Illustration A above is expressed by the numeral 33; that is, there are 3 sets of ten and 3 sets of one (or one set of 3) members. Note that in the numeral 33 the digit 3 is used twice. The 3 at the right names the number of 3 sets of one; the 3 at the left names the number of 3 sets of ten. The number represented by the digit 3 depends on the place (position) it occupies in the numeral.

When the digits 0 through 9 are used alone they are called one-place, one-digit, or one-figure numerals. When we have a sufficiently large number of objects to think of as grouped into sets of ten and sets of one, we record the number with a two-digit numeral, as in the preceding illustration.

When we have recorded the numbers of sets from the empty set through 9 sets of ten and 9 sets of one, we have exhausted the possibilities of recording whole numbers with numerals that use only ten different symbols and two positions. At this point, we must think of objects as collected into a larger set. We follow the pattern we began by collecting 10 sets of one into 1 set of ten. Now we collect 10 sets of ten into 1 set and name that set one hundred (Figure 2).

Figure 2:



How many sheets of 100 stamps? strips of ten stamps? single stamps?
124 means _?_ hundreds, _?_ tens, _?_ ones.

We follow a similar pattern in dealing with larger numbers of objects, grouping them into sets of ten hundreds (one thousand), ten thousands, ten ten-thousands (one hundred thousands). Each time we must use another place to the left in the numeral that records the number. For example:

a.

$$\begin{array}{r} 9 \\ +1 \\ \hline 10 \end{array} \qquad \begin{array}{r} 99 \\ +\ 1 \\ \hline 100 \end{array} \qquad \begin{array}{r} 999 \\ +\ \ 1 \\ \hline 1000 \end{array} \qquad \begin{array}{r} 9{,}999 \\ +\ \ \ \ 1 \\ \hline 10{,}000 \end{array}$$

b.

$$4{,}357 = 4000 + 300 + 50 + 7 \qquad \begin{array}{r} 4000 \\ 300 \\ 50 \\ 7 \\ \hline 4357 \end{array}$$

An important principle in our numeration system is that of place-value. Place-value simply means that the number a digit represents depends on the place it occupies in the complete numeral.

It will not be an easy task for pupils at this grade level to write and read numerals for large numbers. If you make it clear to them that any three digits in the fourth, fifth, and sixth places respectively name the thousands, they will be aided in recognizing that reading such numerals is no more difficult than reading in the hundreds or units. Do not expect all of your pupils to master reading and writing numerals through six places, but all pupils should understand basic place-value concepts.

TEACHING PROCEDURE

Exploration. Review with your pupils the concept that each position occupied by a numeral written in the decimal system has a name. For example, on the chalkboard write the numeral 647.* Point out that this is a three-place numeral and that each digit has a name and represents a certain value. Ask, "Who would care to analyze the numeral 647 and tell us what it represents?" The responses should be similar to the following:

(1) The numeral 647 is read: six hundred forty-seven.
(2) The 6 in 647 means 600 or 6 hundreds; the 4 in 647 means 40 or 4 tens; and the 7 in 647 means 7 ones.

---

* Perhaps the teacher may wish to explore this type of analysis by starting with a two-place numeral such as 38 instead of a three-place numeral. In either case, use the analysis similar to that mentioned above.

Next, write the numeral 999 on the chalkboard. Lead the pupils to see that 999 is a three-place numeral even though the same digit is used in all three positions. Ask, "Who could analyze the numeral 999 and tell us what it really represents?" Some of the responses might be:

1. There are 9 hundreds, 9 tens, and 9 ones.
2. From left to right, the first 9 means 900; the second 9 means 90; and the third 9 means 9 ones.

On the chalkboard show the following illustration and ask the pupils to make an analysis of each of the numerals. Ask, "Why are the numerals 4,307 and 6 placed as they are?"

ten thousands	thousands	hundreds	tens	ones
3	8,	8	1	3
	4,	3	0	7
			4	9
1	2,	1	5	8
		6	2	0
				6
1	6,	2	0	0

Because 4,307 represents: 4 thousands, 3 hundreds, 0 tens, and 7 ones.

Because 6 represents 6 ones.

Next, ask, "Why are the numbers represented by each of the numerals easy to read?"

1. Because each column has a special position and special name.
2. The chart tells us the name for each position.

Extend the concept of the decimal system by the following illustration, suggesting that the pupils assist by filling in the blank spaces:

10 ones are the same as 1 ten

10 tens are the same as 1 hundred

10 hundreds are the same as 1 thousand

10 thousands are the same as 1 ten thousand

For an analysis of four-place numerals, write the numeral 1,234 on the chalkboard. Ask, "How many hundreds does this numeral represent?" 12 Name two ways in which we can read the numeral 1,234."

1. One thousand; two hundred thirty-four
2. Twelve hundred thirty-four.

To help pupils to further analyze the numeral 1,234 and others, ask the class to write the numerals represented by each of the digits and their arrangement. Illustration (a) should serve as a model:

Illustrations:	(a)	(b)	(c)	(d)
Numeral:	1,234	2,341	3,412	4,123
Meaning:	1,000 200 30 4	2,000 300 40 1	3,000 400 10 2	4,000 100 20 3

Have the pupils take note that each illustration has a different arrangement of the digits 1,234. Thus, because of place-value, a difference in arrangement of the digits changes their meaning. Each example has the same digits but represents a different number.

Next, have the pupils fill in the missing numerals, for the following examples. Have them do these at the chalkboard.

2,000	5,000	8,000	7,000
600	800	300	200
90	20	00	00
5	0	4	0
2,695	5,820	8,304	7,200

SUMMARY OF BASIC IDEAS

1. The place-value scheme that the decimal system utilizes has a base of ten.
2. The importance of place-value is evident especially when we want to express large numbers.
3. The ten basic symbols used by our system of numeration are: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

4. When analyzing place-value we find that 10 ones = 1 ten, that 10 tens = 1 hundred, that 10 hundreds = 1 thousand, and that 10 thousands = 1 ten thousand.

5. The arrangement of a given set of digits in the numeral may change its meaning (value).

Pupil Exercises. Provide each pupil with worksheet No. 6.

Briefly discuss examples 1, 2, and 3 with the pupils to make sure they understand the instructions. Examples 4, 5, and 6 may need additional discussion. Braces are needed for example 4.

In example 6 (a), pupils may not be familiar with using parentheses in number patterns.

Interim Mathematics Unit - Grade 6
San Diego City Schools (1-65)

Worksheet No. 6

READ THE INSTRUCTIONS FOR EACH SET OF EXAMPLES.

1. Write the numeral for the number that means:

   (a) 8 hundreds, 6 tens, 4 ones 864

   (b) 7 hundreds, 0 tens, 9 ones 709

   (c) 6 hundreds, 7 tens, 8 ones 678

2. Fill in each blank space with the correct digit.

   (a) 23 + 0 = 2 tens and 3 ones

   (b) 88 + 3 = 9 tens and 1 one

   (c) 999 + 1 = 10 hundreds, 0 tens, 0 one

   (d) 888 + 11 = 8 hundreds, 9 tens, 9 ones

3. Arrange the following numerals in the correct order from greatest to least:

   (a) 6543, 5463, 3645, 6435, 7654, 8543, 3546

   8543, 7654, 6543, 6435, 5463, 3645, 3546

   (b) 10,123; 9,258; 10,321; 8,958; 10,013; 9,999

   10,321, 10,123, 10,013, 9,999, 9,258, 8,958

   (c) 9807, 9708, 9087, 0798, 9870, 9078

   9870, 9807, 9708, 9087, 9078, 0798

4. List the numerals for the odd numbers beginning with 97 and ending with 107.

   {97, 99, 101, 103, 105, 107}

5. Complete the chart.

   (a) In 2,475, which digit represents hundreds? 4

   (b) In 12,385, which digit represents thousands? 2

   (c) In 48,576, which digit represents tens? 7

   (d) In 12,439, which digit represents ten thousands? 1

6. In the blank spaces, write the missing numerals which complete the pattern.

   (a) (5 + 7), (6 + 8), (7 + 9), (8+10), (9+11), (10+12), (11+13)

   (b) 975, 935, 895, 855, 815, 775, 735, 695

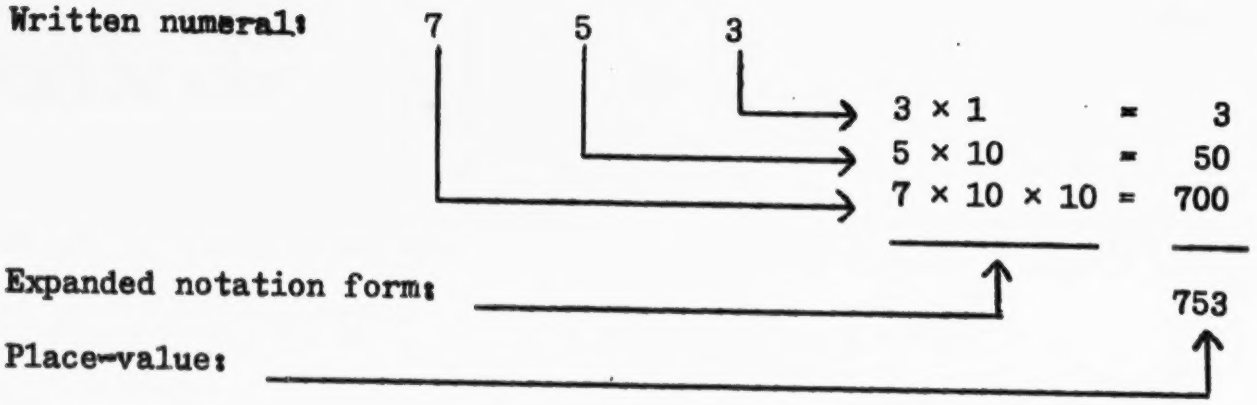
E. Expanded Notation

Objectives. (1) To develop the place-value meaning of a numeral by using the expanded notation form. (2) Provide opportunities to become familiar with a computing device (abacus) which illustrates place-value and permits place-value analysis.

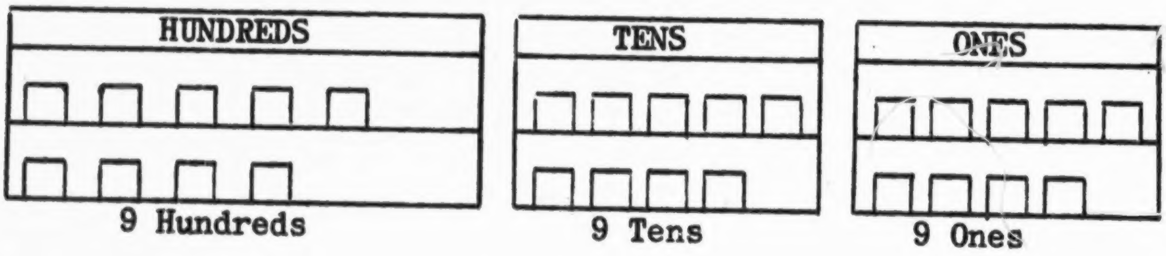
Background Information for the Teacher. We have learned that the distinguishing feature of a positional system as opposed to a non-positional system of numeration is the use of basic numerals whose number values are determined by their position in a sequence of numerals. This is the concept of place-value.

Our decimal system of numeration uses ten basic numerals, or digits (a derivation from the Latin word digitus, meaning finger). It is the use of the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 in a place-value scheme based on the powers of ten that we have our decimal numeration system.

At this point, consider the numeral 753. The position occupied by each digit indicates (from right to left) the number of ones, tens, and hundreds. The position to which reference is made is an actual physical location relative to the ones position (see illustration below). From right to left, however, the number represented by the digit is ten times the value of the number represented by the digit which precedes it.



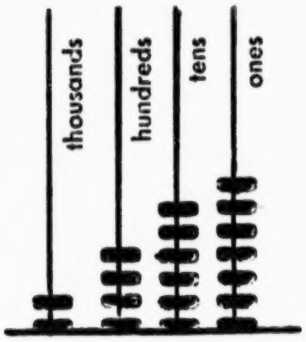
Further analysis of place-value can be achieved through the use of the place-value charts (synonym: pocket charts; place-value card holders) as shown below:



	Expanded Notation						Numeral
	900	+	90	+	9	=	999
or	(9 × 100)	+	(9 × 10)	+	(9 × 1)	=	999
or	(9 × 10 × 10)	+	(9 × 10)	+	(9 × 1)	=	999

The abacus is a device which supplements the use of chalkboard and pocket chart illustrations regarding place-value. An illustration to show the use of an abacus is shown in Figure 1.

Figure 1:



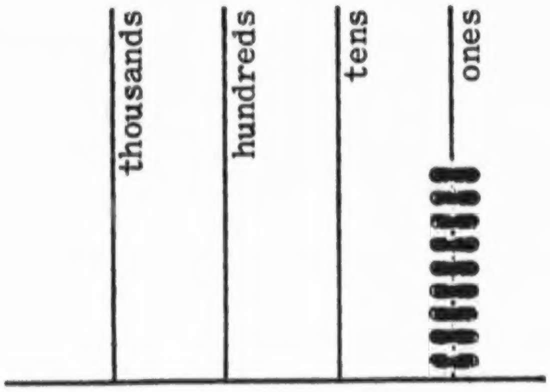
$$\begin{array}{rrrr}
 & & & 1 \\
 & & 10 & 1 \\
 & & 10 & 1 \\
 & 100 & 10 & 1 \\
 & 100 & 10 & 1 \\
1000 & 100 & 10 & 1 \\
1000 & 100 & 10 & 1 \\
\hline
2000 & +400 & +60 & +7 = 2,467
\end{array}$$

From the illustrations listed on page 36, one can readily appreciate our positional system and the orderly way in which numbers can be analyzed and represented.

TEACHING PROCEDURE

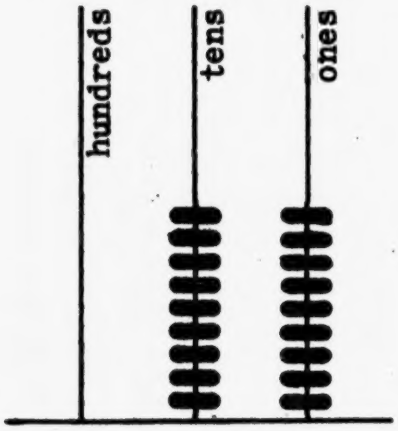
Exploration. A possible way to approach the teaching of expanded notation is to review with your pupils the concept of place-value. Next, introduce them to the abacus and emphasize its usefulness in terms of representing and analyzing numbers. On the chalkboard (or use a model if one is available) use the following illustration (Figure 2):

Figure 2:



Ask, "How many ones are shown in illustration B?" 9 Next, illustrate an abacus which shows the following, illustration (Figure 3):

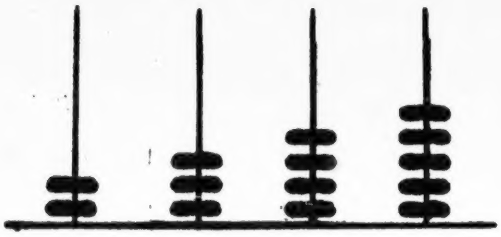
Figure 3:



Ask, "How many tens and ones are shown on the abacus in illustration C?" 9 tens, 9 ones

Now, illustrate an abacus (Figure 4) which shows the following:

Figure 4:



Ask the following questions:

1. How many ones are shown on the ones' wire? 5

2. How many tens are shown on the tens' wire? 4
Write the numeral for 4 tens. 40

3. Each ten is equal to how many ones? 10

4. Then how many ones would it take to equal 4 tens? 40

5. How many hundreds are shown on the hundreds wire? 3
Write the numeral for 3 hundreds. 300

6. Each hundred is equal to how many tens? 10

7. Then how many tens would it take to equal 3 hundreds? 30
How many ones would it take? 300

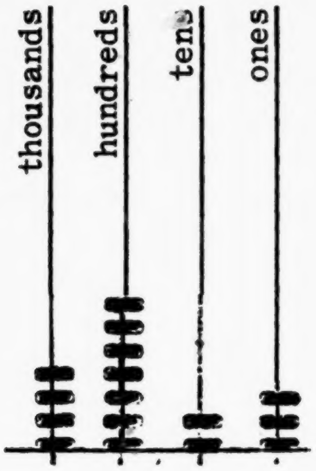
8. How many thousands are shown on the thousands' wire? 2
Write the numeral for 2 thousands. 2000

9. Each thousand is equal to how many hundreds? 10

10. Then how many hundreds would it take to equal 2 thousands? 20
How many tens would it take? 200 How many ones? 2000

To analyze four-digit numerals, represent the following illustration (Figure 5) on the chalkboard:

Figure 5:



Ask the pupils to study the abacus illustration and have them write the numeral which is represented. 4,723 . Use the following questions and direction which are designed to help pupils analyze the place-value concept:

1. How many beads are on the wire in the thousands' place? 4
   Do these beads represent the quantity 4 × 1,000? Yes What is the product of 4 × 1,000? 4,000
2. Write a numeral to show what the beads on the hundreds' wire represent? 700 What is 2 × 10? 20 What is 3 × 1? 3
3. Complete the following sentence represented by the abacus illustration:

4,723 = 4,000 + 700 + 20 + 3

Next, indicate to the pupils that the abacus and the place-value idea permit us to read and express numbers in many different ways. On the chalkboard write the numeral 9,999. Ask, "Who could read this number in two ways?" Responses should be:

1. Nine thousand, nine hundred ninety-nine.
2. Ninety-nine hundred ninety-nine.

Now ask, "When we read it the second way, we are thinking of the 9 thousands as how many hundreds?" 99

Write the following numerals on the chalkboard and provide practice in reading and writing each example:

1,264 One thousand, two hundred sixty-four.
3,589 Three thousand, five hundred eighty-nine.

2,721 Two thousand, seven hundred twenty-one.

8,497 Eight thousand, four hundred ninety-seven.

9,351 Nine thousand, three hundred fifty-one.

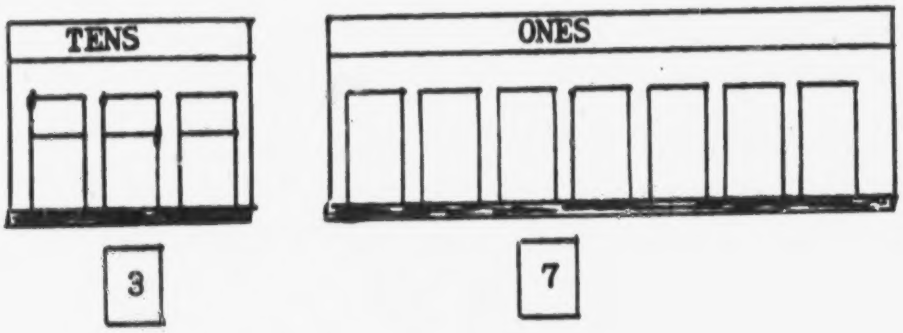
Have some pupils write these at their desks and ask others to write on the chalkboard. Check the correct use of the comma and the hyphen.

Next, on the chalk tray place a set of cards on which the ten digits have been written, that is:*

$\boxed{0}\quad\boxed{1}\quad\boxed{2}\quad\boxed{3}\quad\boxed{4}\quad\boxed{5}\quad\boxed{6}\quad\boxed{7}\quad\boxed{8}\quad\boxed{9}$

Put a set of 37 colored paper strips in a place-value chart hanging above the chalk tray. Have a pupil count by ones to determine the number of the set of colored paper strips. After the pupil (or the class) has concluded that this is a slow method of finding the number of members in the set, have the pupil form as many sets of ten strips of colored paper as he can. Have him use paper clips or rubber bands to bind each set of ten strips together as in the illustration below (Figure 6):

Figure 6:



Now ask someone to choose from the set of cards those which are needed to represent the number of sets of ten and sets of one, that is, cards $\boxed{3}$ and $\boxed{7}$. Emphasize to the class that the 7 digit occupies a place in the numeral called the ones' place and 3 digit occupies the place in the numeral called the tens' place.

Under the pocket chart, below the 3 and the 7 write these symbols:

$$\begin{array}{cc} \boxed{3} & \boxed{7} \\ 3 \times 10 & 7 \times 1 \end{array}$$

---

*This activity can be modified depending on the maturity level and experience with the place-value concept.

Point out to the pupils that by adding the quantity represented by those two sets, we find the whole number we are seeking. For example:

$$\boxed{3} \qquad\qquad \boxed{7}$$

$$3 \times 10 = 30 \quad \text{and} \quad 7 \times 1 = 7$$

$$30 \quad + \quad 7 \quad = 37$$

At this time, indicate to the pupils that only two numbers can be added at a time. A standard symbol must be used to show that numbers are being "grouped" and should be added first. Thus, we use "( )," which we call parentheses. With this new symbol we can show the above example in the following form:

$$\boxed{3} \qquad\qquad \boxed{7}$$

$$(3 \times 10) \quad \text{and} \quad (7 \times 1) =$$

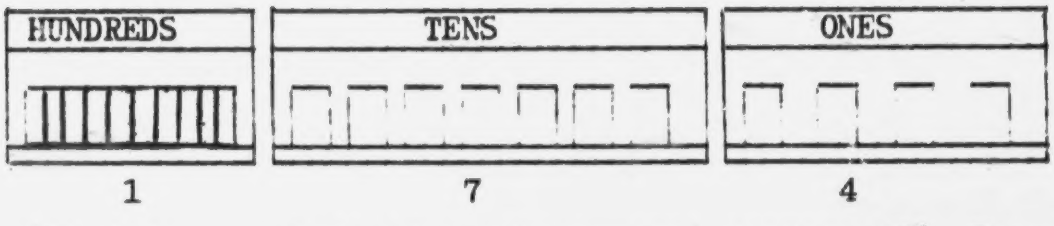
$$30 \quad + \quad 7 \quad = \quad 37$$

Now, remind the pupils that 37 is the numeral which represents the expanded form 30 + 7.

Continue with three or four similar examples, using a set of twenty-nine strips, a set of fifty-three strips, and a set of eighty-nine strips.

Next, place seventeen sets of ten strips in the pocket chart. Have someone count the number of sets of ten to determine the number of strips in this set. Lead the class to conclude that, although this is much quicker than counting by ones, there is a more efficient method that can be used, that is, group 10 sets of ten to form 1 set of one hundred. Thus, we now have (See Figure 7) the numeral 174.

Figure 7:



Now, ask, "What does each digit in the numeral represent?" Responses should be similar to:

(1) The digit one (1) represents one set of one hundred.

(2) The digit seven (7) represents seven sets of ten.

(3) The digit four (4) represents four sets of one.

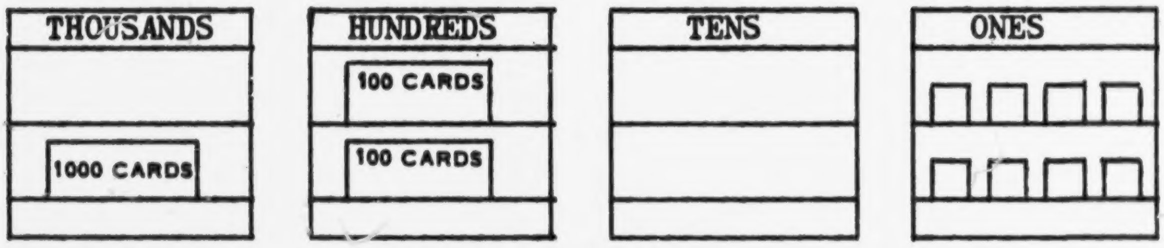
Again explain that other symbols can be used to show the number of hundreds, tens, ones, that is:

$$\overset{\textcircled{1}}{(1 \times 100)} + \overset{\textcircled{7}}{(7 \times 10)} + \overset{\textcircled{4}}{(4 \times 1)}$$

$$100 + 70 + 4 = 174$$

When dealing with thousands, use an example similar to that of the one below. Again explain that by using expanded notation, we can find the number as shown in the illustration below. Have the pupils note the empty set of TENS and the way it appears in the expanded notation form (Figure 8).

Figure 8:



$$(1 \times 1000) + (2 \times 100) + (0 \times 10) + (8 \times 1)$$

$$1{,}000 + 200 + 0 + 8 = 1{,}208$$

An example which the pupils will need to analyze might be number 555. Write the number 555 on the chalkboard. Have a pupil write the place-value meaning of this symbol under the given numeral. For example:

$$555$$

$$(5 \times 100) + (5 \times 10) + (5 \times 1)$$

$$500 + 50 + 5$$

Ask the pupils to respond to the following question: "Who can make a general statement about the digit 5 in each position?" The responses should be similar to the following:

1. The digit 5 on the right indicates that there are 5 sets, each containing one member.
2. The digit 5 in the middle also indicates five sets, each of which contains one member.

3. The digit 5 at the left also indicates five sets, each of which contains one member.

Guide the pupils to conclude that as we move from right to left, each position in the numeral is associated with sets ten times as large as those associated with the previous position.

Present other numerals in which a single digit is repeated three times. Have pupils come to the chalkboard to write the place-value meaning of each symbol as well as the expanded notation. Guide your pupils to generalize that a single digit can represent many different numbers, depending upon the position the digit holds in a numeral.

Write numerals containing the digit 0 in various positions such as 560 and 506. In each instance ask pupils to indicate the place-value meaning of the decimal numeral and to write the expanded notation for the given numeral. For example:

$$\begin{aligned} 560 &= (5 \times 100) + (6 \times 10) + (0 \times 1) \\ &= 500 + 60 + 0 \end{aligned}$$

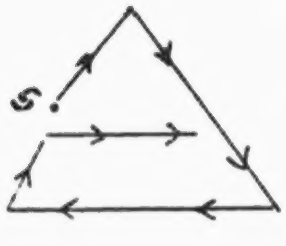
$$\begin{aligned} 506 &= (5 \times 100) + (0 \times 10) + (6 \times 1) \\ &= 500 + 0 + 6 \end{aligned}$$

Pupil Exercises. Provide each pupil with worksheet No. 7.

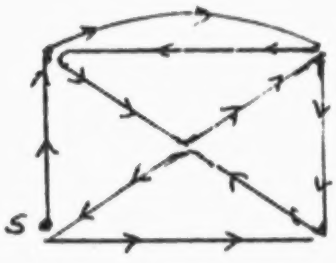
Since many pupils are not familiar with this type of analysis and notation, it is suggested that similar examples be written on the chalkboard for group practice prior to doing the worksheet independently.

The "Just for Fun" examples are optional. If time permits, these examples may be used as a basis for group discussion. Possible approaches to (a), (b), and (d) are given below with "s" denoting the starting point; (c) is impossible.

(a)



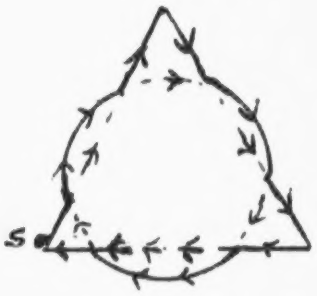
(b)



(c)

Impossible

(d)



STUDY EXAMPLE 1 BEFORE STARTING THE FOLLOWING EXERCISES.

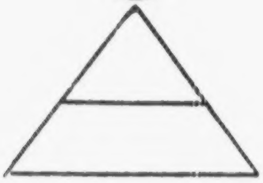
	Numeral	Meaning	Expanded Notation
1.	47	4 tens and 7 ones	40 + 7
2.	89	8 tens and 9 ones	80 + 9
3.	207	2 hundreds, 0 tens, and 7 ones	200 + 0 + 7
4.	536	5 hundreds, 3 tens, and 6 ones	500 + 30 + 6
5.	4,320	4 thousands, 3 hundreds, 2 tens, and 0 ones	4000 + 300 + 20

STUDY EXAMPLE 6 BEFORE STARTING THE FOLLOWING EXERCISES. COMPLETE THE FOLLOWING:

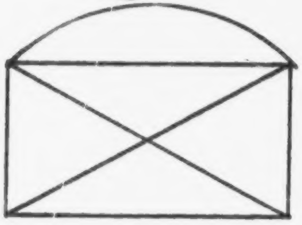
6.	68	(6 × 10) + (8 × 1)	60 + 8
7.	75	(7 × 10) + (5 × 1)	70 + 5
8.	90	(9 × 10) + (0 × 1)	90 + 0
9.	125	(1 × 100) + (2 × 10) + (5 × 1)	100 + 20 + 5
10.	423	(4 × 100) + (2 × 10) + (3 × 1)	400 + 20 + 3
11.	769	(7 × 100) + (6 × 10) + (9 × 1)	700 + 60 + 9
12.	2,570	(2 × 1000) + (5 × 100) + (7 × 10) + (0 × 1)	2000 + 500 + 70

JUST FOR FUN. CIRCLE THE LETTER OF EACH DIAGRAM WHICH MAY BE DRAWN WITHOUT LIFTING YOUR PENCIL FROM THE PAPER AND WITHOUT TRACING A PREVIOUSLY DRAWN LINE. (Show your work.)

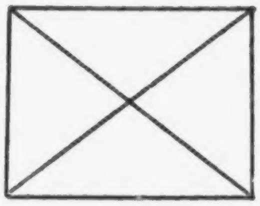
[✓ (a)]



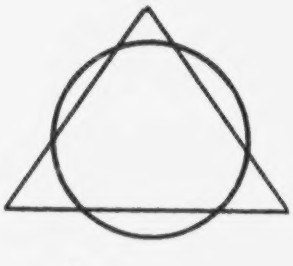
[✓ (b)]



(c)



[✓ (d)]



F. Renaming Numbers (Thinking about numbers in different ways)

Objectives. (1) To help pupils understand that the structure of our numeration system permits thinking about and naming numbers in several ways. (2) To develop skill in renaming numbers which is vital to understanding the "regrouping" idea used in the basic operations: addition, subtraction, multiplication, and division.

Background Information for the Teacher. The idea that a number has many names was introduced in the section entitled "Number-Numeral Distinction," page 21. With this understanding of place-value and the use of such devices as place-value charts and the abacus, it is now possible to develop this concept in greater depth.

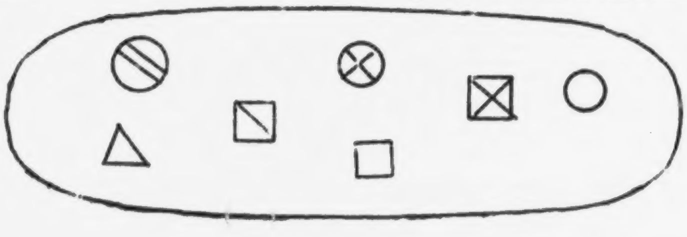
Consider the number 7 as an example; this can be shown as: 3 and 4, seven, (8 - 1), (1 × 7), (2 + 2 + 3), VII, etc. When dealing with a two-digit numeral such as 58, the following possibilities may be considered: 5 tens and 8 ones, 4 tens and 18 ones, 40 ones plus 18 ones, 38 and 20, etc. Three-digit, four-digit, and other numbers lend themselves to the same pattern. For example, the number 4,605 can be renamed in many ways using ones, tens, hundreds, and thousands:

4,605 = 4,605 ones
4,605 = 46 hundreds + 0 tens + 5 ones
4,605 = 4 thousands + 6 hundreds + 5 ones
4,605 = 460 tens + 5 ones
4,605 = 4,000 + 600 + 5
4,605 = 4,600 + 5
4,605 = 4,500 + 100 + 5
4,605 = 4,500 + 90 + 15

The last two names for 4,605 are especially useful in the understanding of addition and subtraction.

TEACHING PROCEDURE

Exploration. Show the following illustration on the chalkboard:



Next, ask someone to come to the chalkboard and write the numeral which represents the number of objects in the illustration. Of course, the numeral 7 names the number of objects represented. Ask the pupils if they will accept the following as names for the same illustration, also:

Seven
6 + 1
8 - 1
5 + 2
5 ones + 2 ones
VII

This should lead to a discussion and provide an opportunity for three-digit numbers. For example, on the chalkboard represent the following chart and numerals and have pupils write different names for the same number. Responses might be:

7	12	16	58
seven	10 and 2	8 and 8	5 tens + 8 ones
VII	6 times 2	sixteen	4 tens + 18 ones
5 and 2	12 ones	1 ten and 6 ones	38 and 20
7 ones	dozen	XVI	50 and 8
(6+1)	(13-1)	(2x8)	(20+20+10+8)

Next, use place-value charts, an abacus, bundles of sticks, and/or other devices to explore ways of naming numbers with three-digit numerals. Emphasis should be placed on finding more than one name for these numbers. The number 243, for example, may be expressed as:

243 = two hundreds, four tens, and three ones
243 = two hundreds, three tens, and thirteen ones
243 = one hundred, thirteen tens, and thirteen ones

After pupils become familiar with different ways of expressing numbers, ask them to find another name for such forms as: seven tens and eleven ones: [illegible]; twenty-two tens and thirteen ones: 233; etc.

Inclusion of such groupings as three hundreds, twelve tens, and eighteen ones: 438; etc., is advisable because the understanding of these ideas are necessary for regrouping in addition and subtraction.

It is important that pupils develop an understanding of these ideas, especially through the use of manipulative materials. A great deal of group discussion and exploration of the numerous possibilities of different names for the same number should be undertaken before individual written exercises are attempted.

If more examples are needed for the development of a better understanding, use the following examples and have pupils come to the chalkboard to fill in the blank spaces or record their responses as they are given:

225 = 2 hundreds + 2 tens + 5 ones
225 = 22 tens + 5 ones
225 = 225 ones
225 = 1 hundred + 12 tens + 5 ones
225 = 2 hundreds + 1 ten + 15 ones
*225 = 200 + 20 + 5
*225 = 200 + 15 + 10

* Answers will vary.

Pupil Exercises. Provide each pupil with worksheet No. 8.

With the more able group, limit the explanation of this worksheet. However, with the less able group, offer as much explanation as you feel is necessary, especially with the examples requiring spelling.

Interim Mathematics Unit - Grade 6
San Diego City Schools (1-65)

Worksheet No. 8

WRITE "true" OR "false" IN EACH BLANK SPACE.

1. 75 may be renamed as 7 tens + 5 ones. true

2. 93 may be renamed as 8 tens + 13 ones. true

3. 104 may be renamed as 1 hundred + 10 tens + 4 ones. false

4. 432 may be renamed as 3 hundreds + 13 tens + 2 ones. true

5. 617 may be renamed as 5 hundreds + 9 tens + 27 ones. true

WRITE THE CORRECT NUMERAL IN EACH BLANK SPACE.

6. 23 tens + 8 ones = 238

7. Fourteen tens + six ones = 146

8. 5 hundreds + 9 tens + 14 ones = 604

9. 9 hundreds + 2 tens + 33 ones = 953

10. 1 thousand + 4 hundreds + 12 tens + 2 ones = 1,522

COMPLETE THE FOLLOWING SENTENCES:

11. 184 = 1 hundred + 8 tens + 4 ones

12. 453 = 3 hundreds + 15 tens + 3 ones

13. 630 = 5 hundreds + 13 tens + 0 ones

14. 705 = 5 hundreds + 10 tens + 105 ones

15. 4,666= 4 thousands + 6 hundreds + 5 tens + 16 ones

WRITE THE WORD NAME FOR EACH OF THE FOLLOWING:

16. 84 eighty-four

17. 75 seventy-five

18. 40 forty

19. 204 two hundred four

20. 1,565 one thousand, five hundred sixty-five

or fifteen hundred sixty-five

G. Number Sentences

Objectives. (1) To help pupils understand the need for number sentences and (2) to develop skill in describing a problem in their own words and then to state the relationship by use of a mathematical sentence.

Exploration. During the language arts period, pupils use words to write sentences which express ideas, feelings, descriptions, etc. In mathematics, numerals and various symbols are used to express ideas, that is, a number sentence (synonym: mathematical sentence; mathematical equation; equation).

Number sentences tell something about numbers and the action related to those numbers. For example, $6 + 7 = 13$ indicates that the numbers 6 and 7 are being operated on by addition; the result being 13. In other words, it is a sentence about 6, 7, 13 and the operation of addition.

When writing number sentences such as $6 + 7 = 13$, we are expressing the idea that both, $6 + 7$ and 13 are different names for the same number. Other names for 13 may be: $9 + 4$; $18 - 5$; $(2 \times 6) + 1$; etc.

Number sentences may be true or false. Examples of different kinds of sentences are shown below:

True sentences: $8 + 7 = 15$; $4 \times 5 = 20$; $(3 + 3) \times 2 = 12$

False sentences: $3 + 7 = 11$; $11 \times 2 = 23$; $(3 + 4) \times 0 = 7$

Open sentences: $4 + 7 = \square$; $19 - \square = 6$; $\square \times 2 = 6$
(Neither true nor false)

When a pupil learns to describe a problem in his own words and then states the relationship in a number sentence form, he probably has a sound approach to problem solving.

TEACHING PROCEDURE

Exploration. Write the following sentence on the chalkboard:

> "Twenty-seven girls and fourteen boys spent an afternoon at the San Diego Zoo studying lions and tigers. How many children were involved in this study?"

Next, indicate to the pupils that words and sentences were used to describe an idea involving sets of girls, boys, numbers, etc. To solve the problem, we would write:

$$27 + 14 = 41$$

It should be emphasized that when combining sets we do not add the objects (boys and girls); we add the numbers associated with the sets.

Point out that numerals and other symbols can be used to express number ideas; this method is the number sentence. Ask, "How can we express this story and solve the problem using a number sentence?" Response might be:

$27 + 14 = 41$	$27 + 14 = ?$
$14 + 27 = \boxed{}$	$27 + 14 = n$
$\Diamond = 14 + 27$	$\boxed{} = 27 + 14$

Encourage a variety of responses, many of which will serve as an introduction to number sentences. Next, indicate to the pupils that sometimes more than two numbers are mentioned in a word sentence. For example, write the following sentence on the chalkboard:

> "Three boys went skin diving near the La Jolla Cove. One boy speared 3 fish, another speared 5 fish, and the third boy speared only 1 fish. How many fish were caught by the three boys?"

Now ask, "How can we state this problem in number sentence form?" Some responses might be:

$$3 + 5 + 1 = \boxed{}$$

$$3 + 5 + 1 = 9$$

If pupils do not give examples using parentheses such as $(3 + 5) + 1 = 9$ or $3 + (5 + 1) = 9$, review with them how, when, and why they are used. For example, the teacher might point out that since only two numbers can be added (this also applies to subtraction, multiplication, and division) at any one time, parentheses are used to indicate which numbers are being operated on first. Parentheses are important to the understanding of the "grouping principle" or associative principle of addition and multiplication. (The phrase associative principle need not be used with sixth-grade pupils.)

Give pupils an opportunity to work on the chalkboard or have them copy and practice some of the following examples of number sentences: *

	(a)	(b)	(c)
(1)	$24 + 17 = \underline{41}$	$13 + 14 + 17 = \underline{44}$	$(2 + 6) + 10 = \underline{18}$
(2)	$\underline{27} + 12 = \underline{39}$	$\boxed{12} + 5 + 20 = 37$	$(12 - 4) + 7 = \boxed{15}$

---

*It is common practice to write the missing numeral in the blank space or within the frame in order to complete each number sentence. However, whenever the pupil is requested to replace the frame with the missing numeral, for example, $\boxed{} + 14 = 20$, he should write his response as: $6 + 14 = 20$ (Do not require him to rewrite the example with the frame and then have him write the missing numeral in this frame.)

(3) $25 - \boxed{13} = 12$ $\quad 37 + \boxed{13} + 10 = 60$ $\quad (\boxed{10} + 0) + 5 = 15$

(4) $29 - 17 = \underline{12}$ $\quad 34 + 10 + \underline{6} = 50$ $\quad \boxed{9} + (7 + 3) = 19$

After the examples are completed by most pupils, call on them to give their solution to each of the examples. This can be a good learning situation for the pupils and an opportunity for the teacher to emphasize certain important points, that is: (a) restating problems as number sentences, (b) in solving number sentences you may only work with two numbers at a time, (c) number sentences can be shown using different forms, and (d) the operation in parenthesis should be completed first.

The inequality symbol, < , (meaning "less than") is introduced in the grade 4 textbook and the inequality symbol, > , (meaning "greater than") is introduced in grade 6. This concept of inequality has a variety of uses and is an important part of the arithmetic program.

Often a pupil's earliest introduction to an idea of numbers is through inequalities. For example, a preschool child soon will recognize whether a playmate has a greater number of toys than he has although neither child may be able to count. Later he may want to know: (a) Is the number of pieces of candy greater than or less than the number of playmates waiting to eat the candy? (b) Is the amount of money saved at least enough to make the purchase? (c) Is the score made by the team greater than or less than that required to win the game?

Before introducing the symbol for "less than," indicate to the pupils that, for example, "5 × 7" and "35" are two names for the same number. The relation symbol, = , used in the number sentence $5 \times 7 = 35$, shows this relationship. Since 5 × 7 "equals" 35, it must be "less than," for example, 36. To show this sentence, write the following inequality sentence on the chalkboard:

$$5 \times 7 < 36$$

Point out that the sentence is read "5 times 7 is less than 36."

Write several examples on the chalkboard such as illustrations (a), (b), and (c) shown below:

(a)	(b)	(c)
$7 \times 9 < 65$	$8 \times 6 < 65$	$\square \times 4 < 21$

Regarding each illustration ask, "In illustration (a) are there 7 nines in 65? Yes Is the sentence true? Yes In illustration (b) what is the largest number of 6's that are contained in 49? 8 Is the sentence true? Yes In illustration (c) what is the numeral for the largest number you can use to replace the frame and make the sentence true?" 5

(In the sentence $\square \times 4 < 21$, the numerals 0, 1, 2, 3, and 4 can also be used as replacements for the frame with the result that the numbers named will make a true sentence. But, only 5 is the largest number that will still yield a true sentence.)

Use other examples to develop the inequality concept. After a variety of illustrations, including the number line, write the following on the chalkboard. Have the pupils copy each example and have them replace the frame (or write within the frame) with the numeral for the largest number that will make the sentence true.

	(a)	(b)
(1)	$\boxed{4} \times 2 < 9$	$\boxed{6} \times 9 < 58$
(2)	$\boxed{3} \times 5 < 17$	$\boxed{5} \times 9 < 50$
(3)	$\boxed{8} \times 6 < 53$	$\boxed{6} \times 7 < 47$
(4)	$\boxed{5} \times 8 < 47$	$\boxed{8} \times 4 < 35$
(5)	$\boxed{9} \times 9 < 88$	$\boxed{9} \times 7 < 69$

Next, introduce the symbol for "greater than," indicating to the pupils that, for example, "5 × 9" and "45" are two names for the same number. Since 5 × 9 "equals" 45, it must be "greater than," for example, 44. To show this sentence, write the following inequality sentence on the chalkboard:

$$5 \times 9 > 44$$

Point out that the sentence is read "5 times 9 is greater than 44."

Write several examples on the chalkboard such as illustrations (a), (b), and (c) shown below:

(a)	(b)	(c)
$9 \times 7 > 61$	$8 + 5 > 12$	$\boxed{5} \times 8 > 36$

In illustration (c), the members of {6, 7, 8, 9, ...} may also be used to make the sentence true. However, the numerals 0, 1, 2, 3, and 4 cannot be used as replacements for the $\boxed{}$ because the result would not be a true sentence.

Use other examples to develop the inequality concept. After a variety of illustrations, including the number line, write the following on the chalkboard. Have pupils copy each example and have them replace the frame (or write within the frame) with the numeral for the smallest number that will make the sentence true.

	(a)	(b)	(c)
(1)	$\boxed{5} \times 7 > 34$	$8 + \boxed{57} > 64$	$\boxed{20} \times 6 > 119$
(2)	$8 \times \boxed{7} > 59$	$6 \times \boxed{4} > 23$	$88 + \boxed{12} > 99$

Pupil Exercises. Provide each pupil with worksheet No. 9.

Unless the group is advanced, read the inequality examples 6 through 9 with the pupils and have them respond orally. Then have them do the exercises independently.

If the less able pupils need help with the "word problems," offer as much help as you feel is necessary.

In examples 10 through 13, some pupils may be interested in discussing why it is necessary to have the agreement of doing the work in parentheses first.

WRITE THE MISSING NUMERAL FOR EACH EXAMPLE.

	(a)	(b)
1.	$35 + 19 = \boxed{54}$	$6 + 12 + 18 = \boxed{36}$
2.	$28 + 42 = \boxed{70}$	$23 + 17 + 40 = \boxed{80}$
3.	$\boxed{9} + 26 = 35$	$\triangle 36 + 14 + 10 = 60$
4.	$33 + \triangle 27 = 60$	$40 + \boxed{45} = 85$
5.	$45 + 15 + \triangle 10 = 70$	$\triangle 7 + \triangle 7 + 14 = 28$

WRITE THE NUMERAL FOR THE LARGEST NUMBER THAT WILL MAKE THE SENTENCE TRUE.

6.	$\boxed{8} \times 5 < 42$	$\boxed{4} \times 6 < 28$
7.	$\boxed{4} \times 15 < 65$	$\boxed{7} \times 10 < 79$
8.	$\boxed{7} \times 20 < 153$	$\boxed{9} \times 50 < 485$

DRAW A CIRCLE AROUND THE SENTENCES WHICH ARE TRUE.

9. [✓ (a)] $9 \times 5 > 44$ [✓ (d)] $(4 + 5) < (7 + 3)$ [✓ (g)] $(27 + 13) = (13 + 27)$
[✓ (b)] $6 + 8 + 14 = 28$ (e) $9 \times 3 > 28$ [✓ (h)] $9 \times 9 > 80$
[✓ (c)] $6 \times 3 < 19$ (f) $(3 \times 5) > (4 \times 4)$ (i) $88 + 13 = 105$

COMPLETE EACH OF THE FOLLOWING. REMEMBER THAT IN NUMBER SENTENCES THE WORK IN PARENTHESES MUST BE COMPLETED FIRST.

10.	$(15 + 25) + 12 =$ 52	$(16 + 0) + 14 =$ 30
11.	$17 + (18 + 12) =$ 47	$(7 \times 5) + 5 =$ 40
12.	$(0 + 19) + 21 =$ 40	$(21 + 7) + 6 =$ 9
13.	$(87 + 13) + 45 =$ 145	$19 + (50 + 10) =$ 24

WRITE A COMPLETE NUMBER SENTENCE FOR EACH PROBLEM.*

14. The distance from Bill's house to Westgate Park is 37 miles. How many miles will Bill travel if he makes one round trip?

$37 \times 2 = 74$ or $37 + 37 = 74$

15. A tank car held 3,145 gallons of oil. It now contains 1,253 gallons. How many gallons are needed to fill the tank?

$1{,}253 + \square = 3{,}145;\ \square = 1{,}892$

---

*Number sentences will vary.

H. Grouping by Sixes

Objective. To examine a system of numeration based on grouping by sixes for the purpose of better understanding our base ten numeration system.

Background Information for the Teacher. A study of a numeration system with a base other than ten will assist elementary pupils to better understand the base ten system. They will be able to apply the knowledge they have gained to other numeration systems.

The decimal numeration system, based on groupings of ten, is sometimes called the base ten system. Base ten uses only ten symbols, that is, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Since base ten is a system with a place-value principle, the base of the system determines the number of digits to be used in writing numerals.

In a system which has a base of five, we need but five symbols, that is, 0, 1, 2, 3, 4; in a base six system, we need but six symbols, that is, 0, 1, 2, 3, 4, 5.

TEACHING PROCEDURE

Exploration. Ask the pupils to imagine that the shepherd of ancient times, instead of using all of the fingers of both hands to develop a syste of counting, used only six fingers for counting purposes. Instead of worki with groups of ten, he would have worked with groups of six. What would th system of numeration have in common with base ten? Let's examine and build system which uses only six symbols.

On the chalkboard, show a set of x's (or tally marks) as shown in Figure 1.

Figure 1 xxxxxxxxxxxxxxx

Ask the following questions:

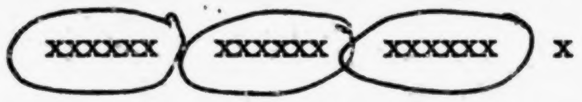
(1) "Can you tell at a glance (without counting) how many marks are shown? No If we count them, can we tell how many there are? Yes How many are there?" 15

(2) "If the 15 marks are grouped into sets of six (Figure 2), can we quickly tell how many there are? Yes How many sets of six x's are there? 2 How many are left over? 3 Then, 15 marks can be grouped to show how many sixes and how many ones?" 2 sixes plus 3 ones

Figure 2 xxxxxx xxxxxx xxx

Next, on the chalkboard, illustrate a total of 19 marks (Figure 3). Ask, "How many sets of six x's are shown? 3 How many are left over? 1 Then 19 marks can be grouped to show how many sixes and how many ones?" 3 sixes and 1 one

Figure 3



For individual practice, illustrate the following on the chalkboard (Figure 4) and have the pupils copy each set of marks. Have them group the x's into as many sets of six as they can and then have them show how many are left over.

Figure 4

(a)	(b)	(c)
xxxxxxx	xxxxx	xxxxxx
xxxxxxx	xxxxxx	xxxxxx
xxx	xxx	xxxxxx

Ask, "How many sixes and how many ones are in example (a)? 2 sixes and 5 ones In example (b)? 2 sixes and 2 ones In example (c)?" 3 sixes and 0 ones

Next, on the chalkboard, provide the pupils with the following practice examples (Figure 5) which require the selection of the largest number of sixes and ones. Have them write the missing numeral in each blank space.

Figure 5

(a) Seven is 1 six and 1 one.
(b) Fourteen is 2 sixes and 2 ones.
(c) Twenty-three is 3 sixes and 5 ones.
(d) Twenty is 3 sixes and 2 ones.
(e) Four is 0 sixes and 4 ones.

Next, illustrate the following chart (Figure 6) on the chalkboard and have pupils take turns completing examples. Have an assortment of objects available in the event a pupil may have difficulty. (In the beginning stages of developing the base six system, use only numbers or objects less than 36.)

Figure 6

	Base Ten Number	Sets of Six	Sixes	Ones
(a)	Eight	xxxxxx xx	1	2
(b)	Fifteen	xxxxxx xxxxxx xxx	2	3
(c)	Twenty-four	xxxxxx xxxxxx xxxxxx xxxxxx	4	0
(d)	Thirteen	xxxxxx xxxxxx x	2	1
(e)	Twenty-seven	xxxxxx xxxxxx xxxxxx xxxxxx xxx	4	3
(f)	Thirty-four	xxxxxx xxxxxx xxxxxx xxxxxx xxxxxx xxxx	5	4

During the above exercise, some of the pupils might discover that the base ten number divided by 6 will yield a quotient (the largest number of sixes) and a remainder (the number of ones). For example, item (a) illustrates 8 ÷ 6 = 1 six and 2 ones; item (b) illustrates 15 ÷ 6 = 2 sixes and 3 ones; item (c) illustrates 24÷ 6 = 4 sixes and 0 ones; etc.

Next, illustrate the following chart (Figure 7) on the chalkboard and have the pupils take turns completing the examples. This time, have them change from grouping by sixes to grouping by fives. (Use objects or numbers less than 25.) Emphasize that the principles used in base 10 and base 6 still apply; however, the grouping differs.

Figure 7

	Base Ten Number	Sets of Five	Fives	Ones
(a)	Seven	xxxxx xx	1	2
(b)	Twelve	xxxxx xxxxx xx	2	2
(c)	Twenty-two	xxxxx xxxxx xxxxx xxxxx xx	4	2
(d)	Nineteen	xxxxx xxxxx xxxxx xxxx	3	4

I. Reading and Writing Base Six Numerals

Objectives. (1) To teach the children base six notation. (2) To illustrate the use of the abacus to represent base six numerals.

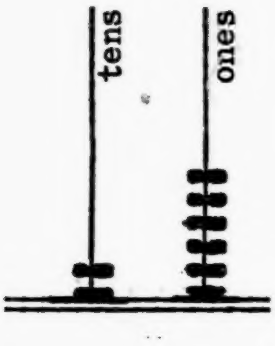
Background Information for the Teacher. Because of the place-value principle of base ten, a numeral based on grouping by tens is a base ten numeral or decimal numeral. For example, 23 means 2 tens and 3 ones. Since it is used daily in our lives, it is written as 23 and not as 23_{ten}. The subscript ten is not required; it is understood.

A system which groups by sixes and uses the place-value principle is called a base six numeral. However, to differentiate between a base ten numeral and a base six numeral, a subscript in the form of a word is written. The word is written to the right and slightly below the numeral which shows that grouping is in sets of six. For example, 23_{six} means 2 sixes and 3 ones.

TEACHING PROCEDURE

Exploration. Briefly review the abacus as a device which is used to illustrate the place-value principle in base ten. (If an abacus is not available, you may illustrate an abacus on the chalkboard.) Next, using the following example (Figure 1), ask these questions:

Figure 1:



(1) "How many wires are needed to show tens and ones?" 2

(2) "How many ones does each disc on the tens' wire represent?" 10

(3) "What base ten numeral is shown on the abacus?" 26

Next, represent the following abacus illustrations (Figures 2, 3, and 4) on the chalkboard which show two wires--a sixes' wire and a ones' wire:

Figure 2

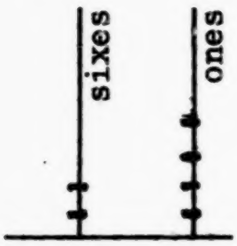


Figure 3

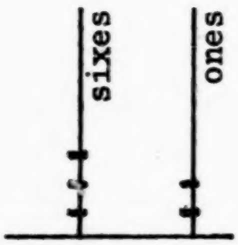
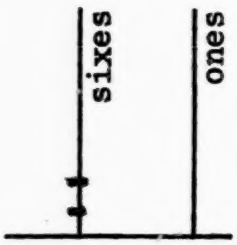


Figure 4



Ask the following questions:

"How many sixes and how many ones are shown by the discs in Figure 2? 2 sixes, 4 ones In Figure 3? 3 sixes, 2 ones In Figure 4?" 2 sixes, 0 ones

At this point, indicate to the pupils that the numeral on the abacus in Figure 2 is written as 24_{six}* and is read "two, four, base six." Figure 3 is written as 32_{six} and is read "three, two, base six." Figure 4 is written as 20_{six} and is read "two, zero, base six."

Encourage discussion of the reason for reading 24_{six} as "two, four, base six." To say "twenty-four, base six" would be incorrect because twenty-four means 2 tens and 4 ones. Continue with the other examples until the pupils see the relationship.

To give pupils practice in reading base six numerals, illustrate the following chart (Figure 5) on the chalkboard:

Figure 5

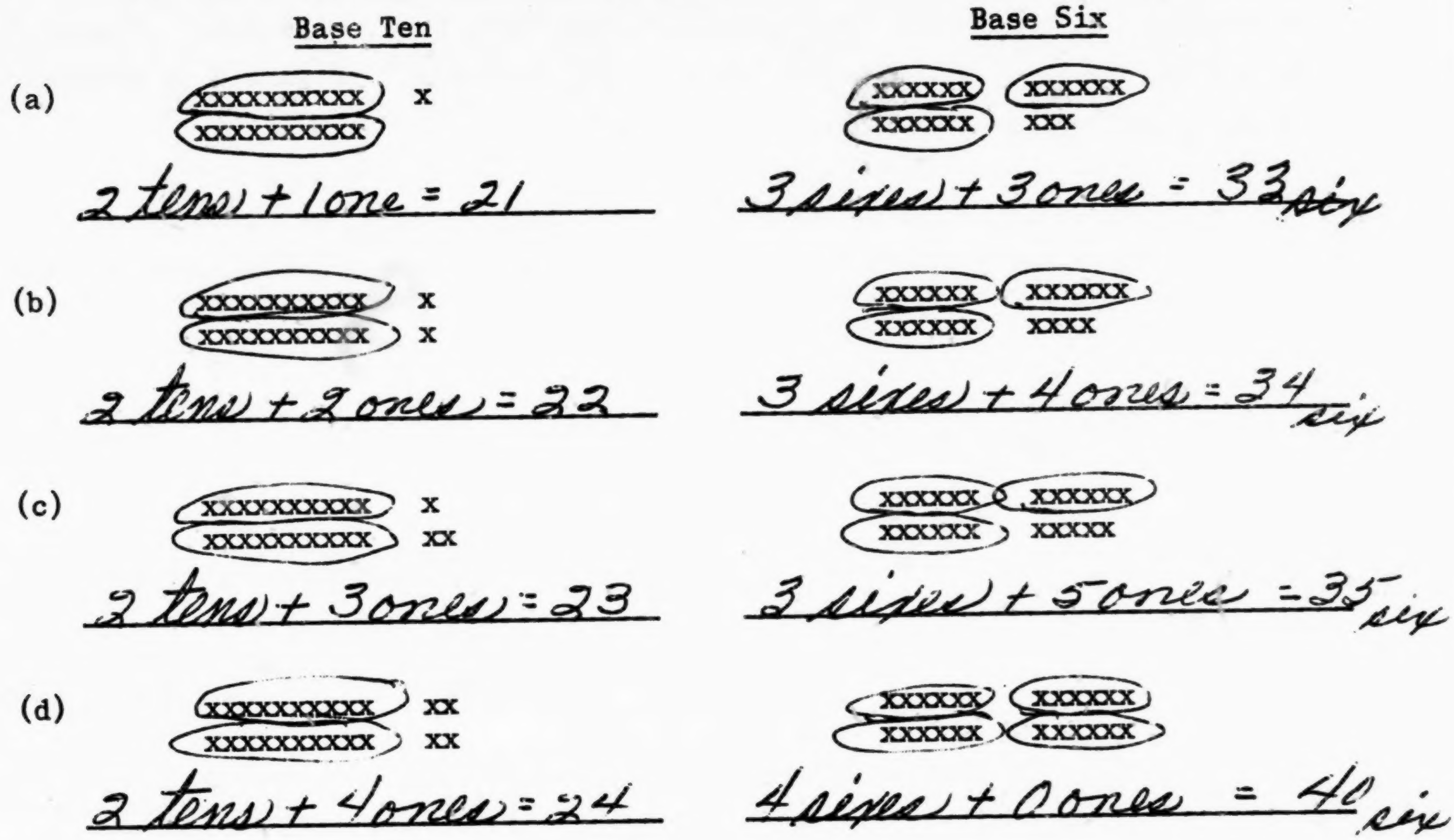
	Objects	Meaning	Base Six Numeral	Reading
(a)	xxxxxx x xxxxxx x	2 sixes and 2 ones	22_{six}	"Two, two, base six"
(b)	xxxxxx xxx	1 six and 3 ones	13_{six}	"One, three, base six"
(c)	xxxxxx x xxxxxx x xxxxxx xx	3 sixes and 4 ones	34_{six}	"Three, four, base six"
(d)	xxxxxx	1 six and 0 ones	10_{six}	"One, zero, base six"
(e)	xxxxxx xxxxxx xxxxxx xxxxxx xxxxxx	5 sixes and 0 ones	50_{six}	"Five, zero, base six"

---

*The subscript six represents the base or grouping by sixes.

To help pupils see the relationship between base 10 numerals and base six numerals, discuss the grouping of 21 objects (tongue depressors, counters, buttons, etc.) into 2 sets of ten and 1 one (Figure 6). Then have them group 21 objects into 3 sets of six and 3 ones. This may be followed by doing the same with 22, 23, and 24 objects.* For example:

Figure 6



(a) 2 tens + 1 one = 21 — 3 sixes + 3 ones = 33_{six}

(b) 2 tens + 2 ones = 22 — 3 sixes + 4 ones = 34_{six}

(c) 2 tens + 3 ones = 23 — 3 sixes + 5 ones = 35_{six}

(d) 2 tens + 4 ones = 24 — 4 sixes + 0 ones = 40_{six}

---

*The use of tally marks, x's, letters, pictures, etc. is optional.

J. Counting in Base Six

Objective. To develop the concept of counting and the number-numeral sequence in base six.

Background Information for the Teacher. The base six numerals that have the number names of zero, one, two, three, four, and five are the same as the base ten numerals for these numbers. The numerals for these names are 0, 1, 2, 3, 4 and 5. Base six does not have the numerals 6, 7, 8 and 9.

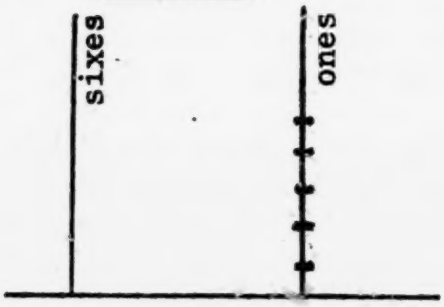
Since grouping is by sixes, the numerals 0, 1, 2, 3, 4 and 5 are adequate to express a number. For example, examine the following chart:

Base Ten Numeral	Base Six Numeral	Base Ten Numeral	Base Six Numeral
1	1_{six} or 1	19	31_{six}
2	2_{six} or 2	20	32_{six}
3	3_{six} or 3	21	33_{six}
4	4_{six} or 4	22	34_{six}
5	5_{six} or 5	23	35_{six}
6	10_{six}	24	40_{six}
7	11_{six}	25	41_{six}
8	12_{six}	26	42_{six}
9	13_{six}	27	43_{six}
10	14_{six}	28	44_{six}
11	15_{six}	29	45_{six}
12	20_{six}	30	50_{six}
13	21_{six}	31	51_{six}
14	22_{six}	32	52_{six}
15	23_{six}	33	53_{six}
16	24_{six}	34	54_{six}
17	25_{six}	35	55_{six}
18	30_{six}	36	60_{six}

TEACHING PROCEDURE

Exploration. Using an abacus or an illustration of an abacus drawn on the chalkboard (Figure 1), ask the following questions:

Figure 1



(1) "What base six numeral is shown on the abacus?" 5

(2) "If another disc is joined to the ones' wire, how many ones would be shown?" 6

(3) "Is there a numeral 6 in the base six system? No What are the basic or standard numerals in base six?" 0, 1, 2, 3, 4, 5

(4) "Can one disc on the sixes' wire replace the six discs on the ones' wire? yes What base six numeral means one six and no ones? 10_{six} Does 10_{six} name the number six in base ten?" yes

(5) "Suppose we join 'one more' disc to the ones' wire. What base six numeral can we use to represent this number? 11_{six} What base ten number does it mean?" 7

Continue the process of joining 'one more' disc to the abacus until the base six numeral 15_{six} is represented. Ask, "What base six numeral means 2 sixes and no ones? 20_{six} What is the base six numeral for twelve?" 20_{six}

As the counting in base six continues and each disc is joined to the abacus to represent a base six numeral, ask the following:

(1) "What is the base six numeral which follows 25_{six}? 30_{six} What base ten numeral does it represent?" 18

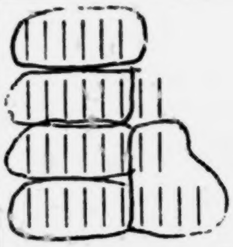
(2) "What is the base six numeral which follows 45_{six}? 50_{six} What base ten numeral does it represent? 30 Name the base six numeral for thirty-five." 55_{six}

To give pupils practice in base six counting, arrange the pupils in line and have each one give his place in line. For example, the first pupil should say "one," the second pupil should say "two," etc. The sixth pupil should say "one, zero," and the twelfth pupil should say "two, zero." Avoid counting more than 35 children during this activity until a base six numeral has been assigned to the numbers 36, 37, 38, etc.

Pupil Exercises. Provide each pupil with worksheet No. 10.

Even though this activity may be relatively easy for some pupils, others will have difficulty, especially the less able pupils. Re-emphasize that one of the purposes of base six (and other base systems) is to review, strengthen, and extend the ideas of place-value. Pupils who gain an understanding of base six probably will understand the work with base ten numerals better because the two systems are similar in many respects.

COMPLETE THE FOLLOWING CHART. THE FIRST ONE IS DONE FOR YOU.

	Group into sets of six	How many sets of six?	How many left over?	Base six numeral	How is the numeral read?
1.	x x x x x x x x	1	2	12_{six}	one, two, base six
2.	r, s, t, u, v w, x, y, z, m a, b, c, d, e f, g	2	5	25_{six}	two, five, base six
3.		5	2	52_{six}	five, two, base six
4.	xxxxxx x x x x x x · x x x x x x · xxxxxx	4	0	40_{six}	four, zero, base six

IN THE ABOVE EXAMPLES, YOU WORKED FROM LEFT TO RIGHT. NOW WORK THE FOLLOWING EXAMPLES FROM RIGHT TO LEFT.

5.	X X X X X X X X X X X X X X X X X X	3	0	30_{six}	three, zero, base six
6.	X X X X X X X X X X X X X X X X X	2	5	25_{six}	two, five, base six
7.	X X X X X X X	1	1	11_{six}	one, one, base six
8.	X X X X X	0	5	5_{six} or 5	five, base six or five

CHAPTER III

OPERATIONS

The developmental ideas in this chapter are devoted to an analysis of the mathematical principles used in basic operations on numbers. The following are considered: (a) analyzing number relationships in a problem defined by a mathematical sentence, (b) deciding on what operation to use, and (c) performing the operation on the numbers.

Pupils who have developed skill in solving problems frequently use many procedures or steps in a different order in solving a variety of problems.

No doubt, pupils have had many experiences with addition, subtraction, multiplication, and division. However, they need to consider the nature of these operations carefully to expand and reinforce the necessary concepts.

It is assumed that this will be the first time pupils will have been formally exposed to such principles as commutative principle of addition, commutative principle of multiplication, associative principle of addition, associative principle of multiplication, and the distributive principle of multiplication over addition.

A. Addition and Subtraction

1. Adding and Subtracting: Tens and Ones

Objective. To develop an understanding of the common algorisms used in addition and subtraction.

Background Information for the Teacher. The development of the meaning basic to the techniques of addition and subtraction is of great importance and should not be slighted. An analysis of the techniques should be given emphasis since they provide a means of understanding the structure of the operations.

If the expression 42 + 37 = ☐ were written on the chalkboard, the sum could be written directly as 79 as in Illustration (a) below. However, if 67 and 58 are the numbers and the operation is addition, the result is 11 tens and 15 ones. This sum,obviously, cannot be written directly as a numeral but may be thought of as 1 hundred, 2 tens, and 5 ones. Then the numeral is written as 125 as in Illustration (b) on the following page:

Illustration (a)

$$\begin{array}{r} 42 \\ +\ 37 \\ \hline 79 \end{array}$$

Illustration (b)*

$$\begin{array}{rcl} 67 & = & 6 \text{ tens},\ \ 7 \text{ ones} \\ +\,\underline{58} & = & \underline{5 \text{ tens},\ \ 8 \text{ ones}} \\ & & 11 \text{ tens},\ 15 \text{ ones} = 125 \end{array}$$

Similarly, when the operation of subtraction is used with numbers whose numerals have more than one digit, the naming of the difference requires careful thinking. For example, if 49 and 23 are the numbers and the operation is subtraction, the difference (unknown addend) can be written directly as the numeral 26 [Illustration (c)]. However, if 32 and 17 are the numbers operated on, the 32 must be thought of as 2 tens and 12 ones. This makes it convenient to subtract 1 ten and 7 ones; the result is then written as 15 [Illustration (d)].

Illustration (c)

$$\begin{array}{r} 49 \\ -\,\underline{23} \\ 26 \end{array}$$

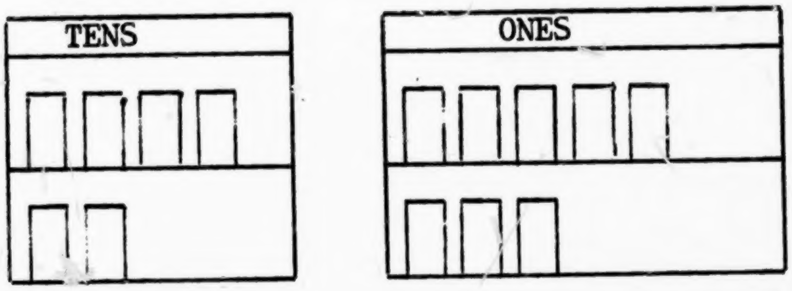
Illustration (d)*

$$\begin{array}{rcl} 32 & = & 2 \text{ tens},\ 12 \text{ ones} \\ -\,\underline{17} & = & \underline{1 \text{ ten},\ \ \ 7 \text{ ones}} \\ & & 1 \text{ ten},\ \ \ 5 \text{ ones} = 15 \end{array}$$

Performing addition and subtraction in this way does not mean that we add or subtract from left to right. In regrouping for addition and subtraction it is convenient to use two numerals rather than one to express a given number. This procedure strengthens the place-value concept.

TEACHING PROCEDURE

Exploration. Using a place-value chart, illustrate an arrangement of strips of colored paper (a chalkboard arrangement of x's will also serve as an illustration), such as:



$$\begin{array}{rcl} 45 & = & 40 + 5 \\ +\,\underline{23} & = & \underline{20 + 3} \\ & = & \_\_ + \_\_ \\ & = & \_\_\_\_ \end{array}$$

Have the pupils study the arrangement so that they can respond to the following questions:

(1) "How many strips are in the first row? 45

Rename the number 45." 4 tens and five ones, or 44+1, 43+2, etc.

---

*The following form may be used for illustrations (b) and (d) above:

(b)

$$\begin{array}{rcl} 67 & = & 6 \text{ tens} + 7 \text{ ones} \\ +\,\underline{58} & = & \underline{5 \text{ tens} + 8 \text{ ones}} \\ & & 11 \text{ tens} + 15 \text{ ones} = 125 \end{array}$$

(d)

$$\begin{array}{rcl} 32 & = & 2 \text{ tens} + 12 \text{ ones} \\ -\,\underline{17} & = & \underline{(1 \text{ ten} \ +\ 7 \text{ ones})} \\ & & 1 \text{ ten} \ +\ 5 \text{ ones} = 15 \end{array}$$

(2) "What is the number in the second row? 23

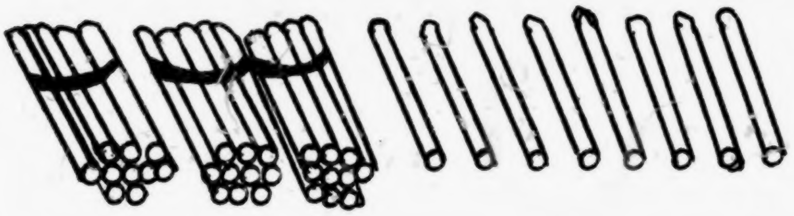
This is how many tens and how many ones?" 2 tens and 3 ones.

(3) "What is the 'shortest' name for 40 + 20? 60

The 'shortest' name for 5 + 3?" 8

(4) "What is the sum of 60 + 8?" 68

After exploring examples such as the one above, introduce the pupils to regrouping to find the sum. For example, using the place-value chart or the chalkboard, represent the following:*



38 = 3 tens, 8 ones

+ 25 = 2 tens, 5 ones

→ 5 tens, 13 ones, or ☐

Point out to the pupils that sometimes when we add, the sum of the ones is more than 9. Then we have to regroup the ones into tens and ones. Ask the pupils to study the above example so that they may answer the following questions:

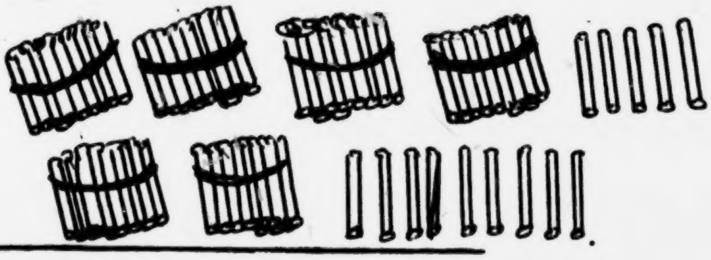
(1) Examine the illustration on the chalkboard. The example using the bundles of sticks shows that we are to add 38 and 25. What is another name for 38? 3 tens and 8 ones. What is another name for 25? 2 tens and 5 ones. What is the sum as shown by the illustration? 5 tens and 13 ones.

(2) Can we regroup 13 ones? Yes, 1 ten and 3 ones.

(3) If we add the 1 ten to the 5 tens, how many tens and how many ones will we have? 6 tens and 3 ones

(4) What is the numeral for 6 tens and 3 ones? 63

Next, use the following example to illustrate the nature of renaming the sum:



45 = ____ tens, ____ ones

+ 29 = ____ tens, ____ ones

→ 6 tens, 14 ones, or ____

---

*Slower pupils will need to work with concrete objects. Milk straws or paper strips may be fastened to the chalkboard with adhesive to illustrate the numbers to be added. A pupil may write the appropriate numerals on the chalkboard beside the straws or strips, while other pupils write on their papers. Take time to develop this regrouping idea. Many experiences will be necessary before all pupils understand the concept.

(2) "Let's investigate. What is 50 - 20? 30 What is 6 - 4? 2
What is 30 + 2? 32 Then what is 56 - 24?" 32

Next, write the following problem on the chalkboard:

> "Several boys put 63 chairs in the auditorium where the fifth grade planned to give a performance. Later, someone removed 26 chairs. How many chairs are left in the auditorium for the performance?"

Ask, "Who can write a number sentence for this problem?" Accept any of the following:

(a) 63 - 26 = □
(b) 63 - 26 = ___
(c) 63 - 26 = △

Now, write the expanded form for the above number sentence so that the pupils will see how the structure of the expanded form permits an analysis of the regrouping concept with respect to subtraction. For example:

63 = 6 tens, 3 ones
- 26 = 2 tens, 6 ones

After pupils have studied the illustration, ask, "Is it possible to subtract 6 ones from 3 ones and get a whole number? no What must we do to perform this operation?" (Guide the pupils to use the regrouping concept, renaming 63 as 5 tens and 13 ones.) * For example:

63 = 6 tens, 3 ones = 5 tens, 13 ones
- 26 = 2 tens, 6 ones = 2 tens, 6 ones

Next, having renamed 63, show the following, completing illustration (a), then showing the form [illustration (b)]:

(a) (b)

63 = 5 tens, 13 ones = 50 + 13
- 26 = 2 tens, 6 ones = 20 + 6
3 tens, 7 ones = 30 + 7 = 37

---

* The teacher may wish to illustrate the minuend 63 with 6 bundles of ten counting sticks and 3 single sticks fastened to the chalkboard or shown in the place-value chart. Let the pupils suggest untying one of the bundles of ten so that there will be 5 tens and 13 ones. Then, let a pupil remove six of the single sticks and 2 of the bundles of ten to illustrate subtracting 26 from 63.

The teacher using this unit is encouraged to illustrate and work several examples with the pupils similar to those shown above. When the teacher feels that the concept has been thoroughly developed and understood, write the following on the chalkboard:

(a)	(b)	(c)	(d)	(e)
81	71	84	90	92
- 57	- 23	- 47	- 53	- 53

While some pupils come to the chalkboard to work examples for practice, have the other pupils work some of the examples at the desks. Have them proceed from the conventional form to the expanded forms. When sufficient mastery has taken place, proceed with the pupil exercises.

Pupil Exercises. Provide each pupil with worksheet No. 11.

A brief explanation of this worksheet will be necessary since a variety of forms are used. Remind the pupils to examine each example; some are addition and some are subtraction.

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Worksheet No. 11

STUDY EACH EXAMPLE CAREFULLY. WRITE THE MISSING NUMERAL IN EACH BLANK SPACE.

1. 73 = 7 tens, 3 ones = 70 + 3
+ 19 = 1 ten, 9 ones = $\boxed{10}$ + 9
8 tens, 12 ones = 80 + 12 = 92

2. 27 = 2 tens, 7 ones = 20 + 7
+ 68 = $\boxed{6}$ tens, 8 ones = $\boxed{60}$ + 8
8 tens, 15 ones = 80 + 15 = 95

3. 63 = 6 tens, 3 ones = 5 tens, 13 ones = 50 + 13
− 49 = 4 tens, 9 ones = 4 tens, 9 ones = −(40 + 9)
= 1 ten, 4 ones = 10 + 4 = 14

4. 81 = $\boxed{8}$ tens, 1 one = 7 tens, 11 ones = 70 + 11
− 58 = 5 tens, $\boxed{8}$ ones = $\boxed{5}$ tens, 8 ones = − (50 + 8)
= 2 tens, 3 ones = 20 + 3 = 23

5.
(a)
30 + 7
+ (40 + 6)
70 + 13 = 83

(b)
20 + 9
+ (60 + 8)
80 + 17 = 97

(c)
60 + 4
+ (20 + 7)
80 + 11 = 91

6.
(a)
87 = 70 + 17
− 49 = − (40 + 9)
= 30 + 8 = 38

(b)
76 = 60 + 16
− 69 = − (60 + 9)
= 0 + 7 = 7

7. (24 + 16) + 19 = 59

50 + (12 + 28) = 90

8. (65 + 25) + 8 = 98

44 + (9 + 11) = 64

9. 45 + 35 = 80

75 − 18 = 57

10. (23 + 12) + (17 + 13) = 65

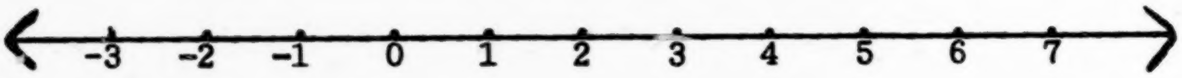
(44 + 16) − 16 = 44

2. Adding and Subtracting: Number Line

Objective. To provide pupils with a graphic device to illustrate the meaning of addition and subtraction and to show the relationship between the two operations.

Background Information for the Teacher. In Chapter I, pages 4-15, pupils were introduced to the number line for the purpose of illustrating the orderly arrangement of our numeration system. In this chapter, emphasis will be on the use of the number line to show the operations.

Since there is no limit to the numbers that may be expressed, the number line should be thought of as extending indefinitely in both directions (see illustration).

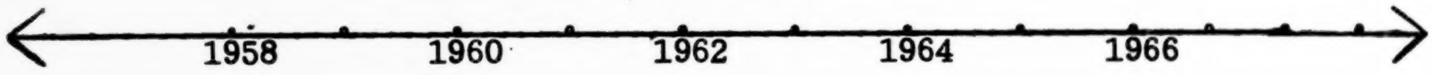


For our purposes, we will be concerned only with the whole numbers, that is, those numbers beginning with 0 and extending to the right, indefinitely. The numbers to the left of 0 are indicated to illustrate that the number line also extends to the left indefinitely.

When choosing a unit of length, use only that unit to mark off points to the right of 0. You may think of this unit of length as an inch, a half-inch, or any arbitrary length, provided the measure is not "too short" or "too long."

In using the number line, one need not always begin with 0. For example, if we are interested in illustrating the years between 1958 and 1964, only that portion which is significant need be represented (Figure 1).

Figure 1

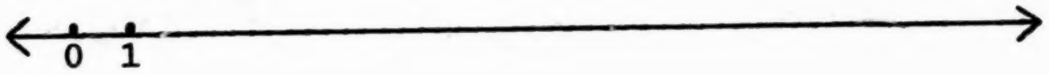


Instruments such as the clock, thermometer, water and gas meters, rulers, protractors, etc. are only few of the many items which use the number line principle. Thus, to look upon the number line as insignificant, contradicts the value of the items mentioned above.

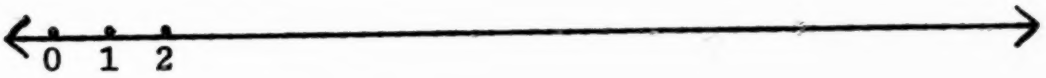
TEACHING PROCEDURE

Exploration. On the chalkboard, draw a picture of a number line or use a number line obtained through a commercial source. Choose a point on the number line to represent the number "0." Refer to this point as "zero" or "point zero."

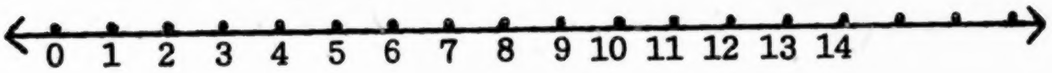
Next, choose a point a short distance to the right of point zero to represent the number 1. Refer to this point as "one" or "point one."



The same distance to the right of point one, mark another point and call it "two" or "point two."



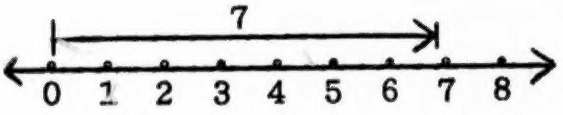
By continuing this process, construct the following:


0
1
2
3
4
5
6
7
8
9
10
11
12
13
14

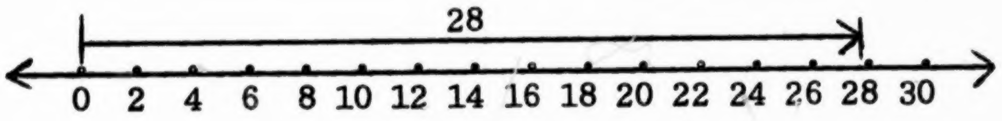
Point out to the pupils that we can show addition (movement to the right) and subtraction (movement to the left) on the number line.

For example:

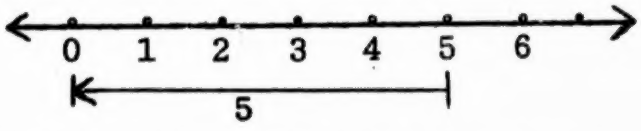
(a) 7 units of length:


7
0
1
2
3
4
5
6
7
8

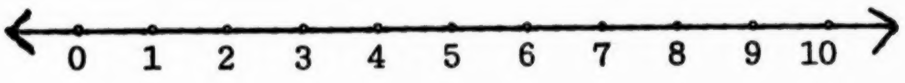
(b) 28 units of length:


28
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30

(c) 5 units of length:



On the chalkboard, represent the following number line:


0
1
2
3
4
5
6
7
8
9
10

(1) "As we look at the number line above, what is the smallest whole number shown?" 0

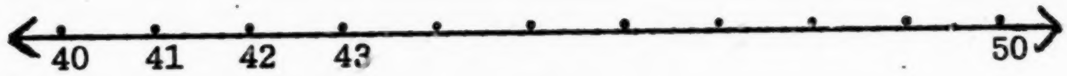
(2) "What do we notice as we move to the right along the number line?"

The numbers increase. The numbers shown are larger.

(3) "Can we tell what whole number will follow the last one shown?" Yes, 11

(4) "What do the arrows on the number line indicate?" We have only a part of the line represented; a line continues in both directions without end.

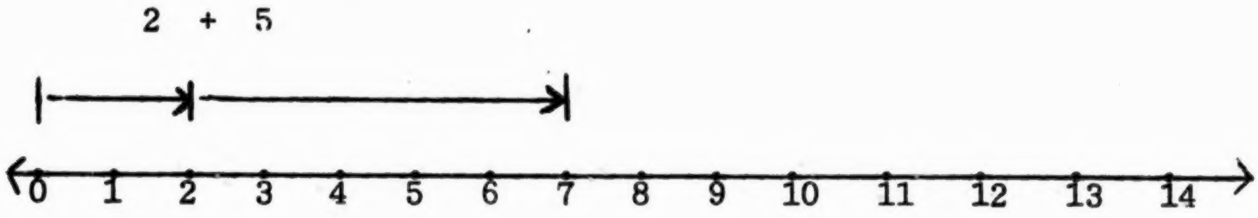
Here is part of a number line showing the whole numbers from 40 through 50.



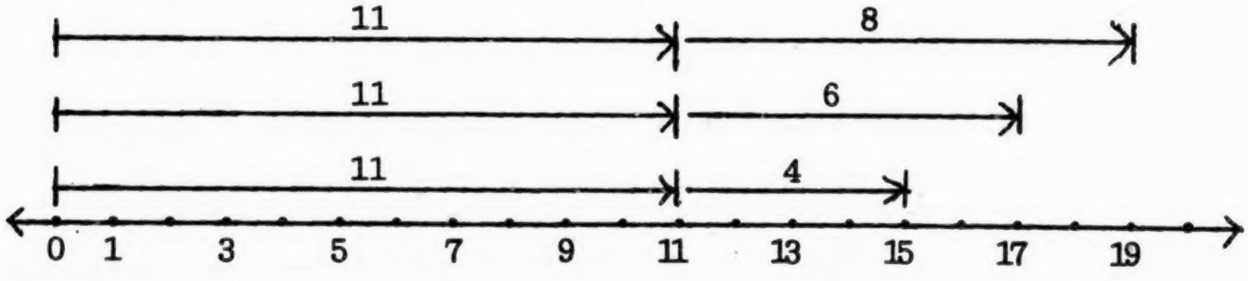
(5) "If we start at point 47 and move to point 42, in which direction do we move?" Left.

(6) "How does the number line show the difference in length from 50 to 46?" 50 is 4 larger than 46 because it is 4 spaces or units to the right of 46.

Next, illustrate to the pupils that the operation of addition can be shown using the number sentence example such as:



Now, point out that adding two-place and one-place numbers is possible, for example:



Ask several pupils to come to the chalkboard and write in the missing numeral for these number sentences and compare them with the above illustration.

(a) 11 + 4 = 15

(b) 11 + 6 = 17

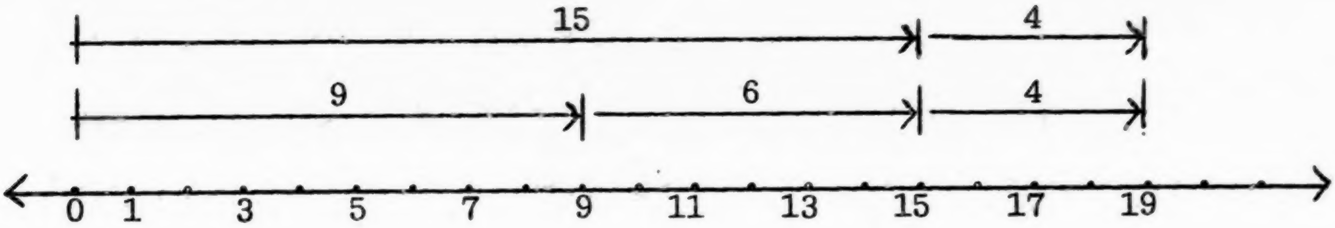
(c) 11 + 8 = 19

Have other pupils, one at a time, come to the chalkboard and illustrate on the number line the following examples:

(a)
22 + 5 = 27
13 + 8 = 21

(b)
19 + 7 = 26
21 + 8 = 29

Following periods of practice with addition, introduce the pupils to higher-decade addition in adding three numbers. Construct the following on the chalkboard:



Have the pupils study the above illustration so that they may answer the following questions:

(1) "Look at the number line. What is the sum of 9 + 6? 15 Are 15 and 9 + 6 two names for the same number? Yes Then, what is the sum of 15 + 4?" 19

(2) "Can we say that (9 + 6) + 4 = 15 + 4?" Yes because they both name the same number.

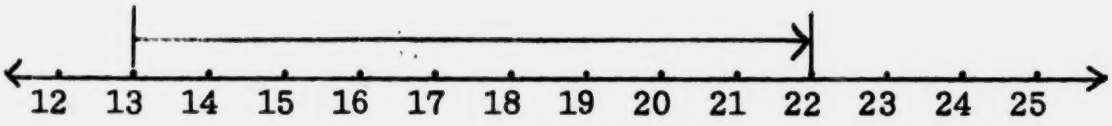
Pupil Exercises. Provide each pupil with worksheet No. 12.

Encourage pupils that accuracy is very important when working examples on the number line. Provide each pupil with a ruler or a straightedge for this activity.

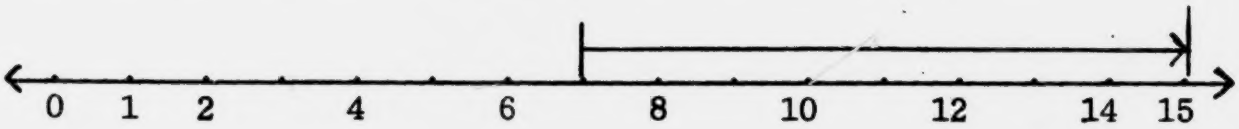
Point out to the pupils that working with whole numbers on the number line will prove useful, especially when working with fractions and decimals.

WORK EACH EXAMPLE ON THE NUMBER LINE, THEN WRITE THE MISSING NUMERAL TO COMPLETE THE SENTENCE.

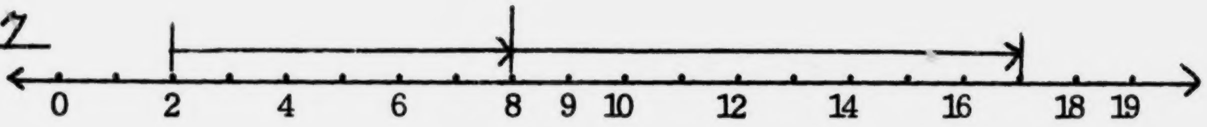
1. 13 + 9 = 22


12
13
14
15
16
17
18
19
20
21
22
23
24
25

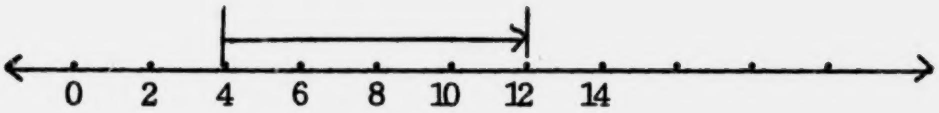
2. 7 + 8 = 15


0
1
2
4
6
8
10
12
14
15

3. (2 + 6) + 9 = 17


0
2
4
6
8
9
10
12
14
16
18
19

4. (4 + 0) + 8 = 12

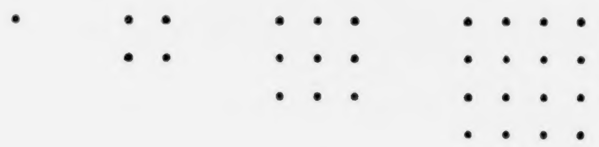

0
2
4
6
8
10
12
14

WRITE THE MISSING NUMERAL FOR EACH EXAMPLE.

	(a)	(b)
5.	63 + 17 = 80	(19 + 18) + 43 = 80
6.	27 + 56 = 83	9 + (27 + 33) = 69
7.	27 + (45 + 0) = 72	(24 + 18) + 46 = 88
8.	37 + $\boxed{13}$ = 50	(△21 + 19) + 30 = 70
9.	△49 + 16 = 65	(33 + 47) + 10 = 90
10.	46 + 27 = 73	(19 + 31) - 31 = 19

DISCOVER THE PATTERN.

11.



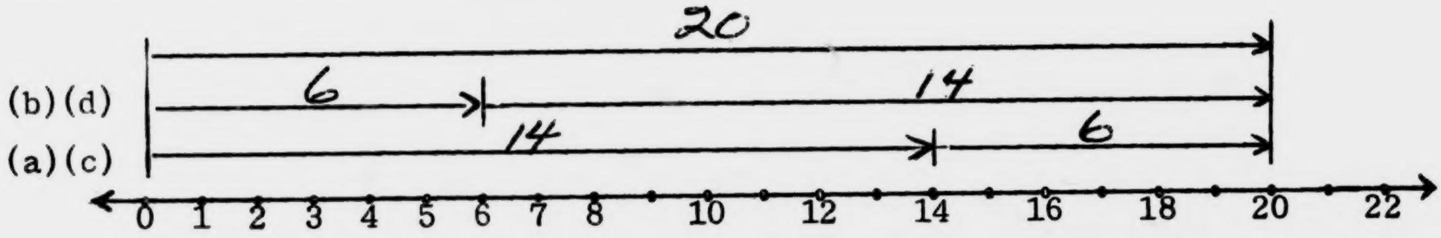
1 4 9 16 25, 36, 49, 64, 81, 100,

To show the operation of subtraction on the number line and the inverse relationship between addition and subtraction, use the following examples for analysis:

Addition

(a)	(b)	(c)	(d)
14 + 6 = 20	6 + 14 = 20	$\begin{array}{r}14\\+\ 6\\\hline 20\end{array}$	$\begin{array}{r}6\\+\ 14\\\hline 20\end{array}$

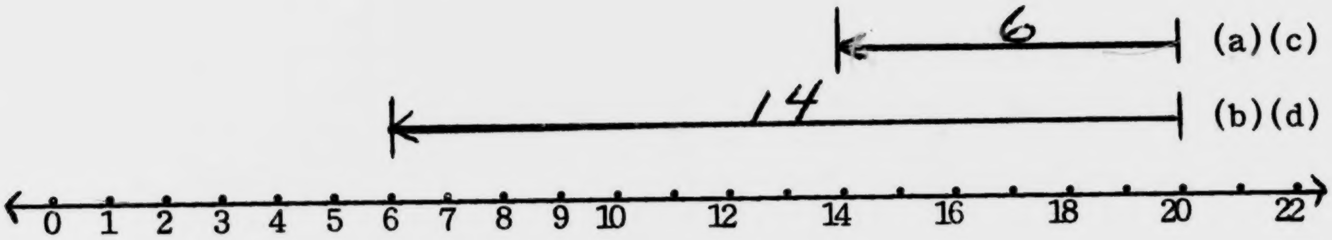
Now, ask someone to come to the chalkboard to show how the addition examples (a),(b),(c), and (d) above, can be shown on the number line. Illustrations should look similar to this:


20
(b)(d)
6
14
(a)(c)
14
6
0
1
2
3
4
5
6
7
8
10
12
14
16
18
20
22

Now, construct a second number line and use the following examples to show the inverse relationship between addition and subtraction:

Subtraction

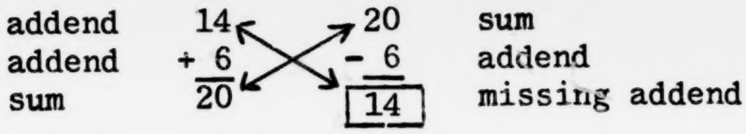
(a)	(b)	(c)	(d)
20 - 6 = 14	20 - 14 = 6	$\begin{array}{r}20\\-\ 6\\\hline 14\end{array}$	$\begin{array}{r}20\\-\ 14\\\hline 6\end{array}$


6
(a)(c)
14
(b)(d)
0
1
2
3
4
5
6
7
8
9
10
12
14
16
18
20
22

Now ask the pupils, "In which direction must we move to show subtraction on the numer line? To the left. Why?"

In addition we move to the right and get a large number we call the sum. In subtraction we move to the left beginning with the sum and finding the missing number or missing addend.

To show another example of the inverse relationship, use the following illustration:


addend
addend
sum
14
+ 6
20
20
- 6
14
sum
addend
missing addend

This may also be written in the form:

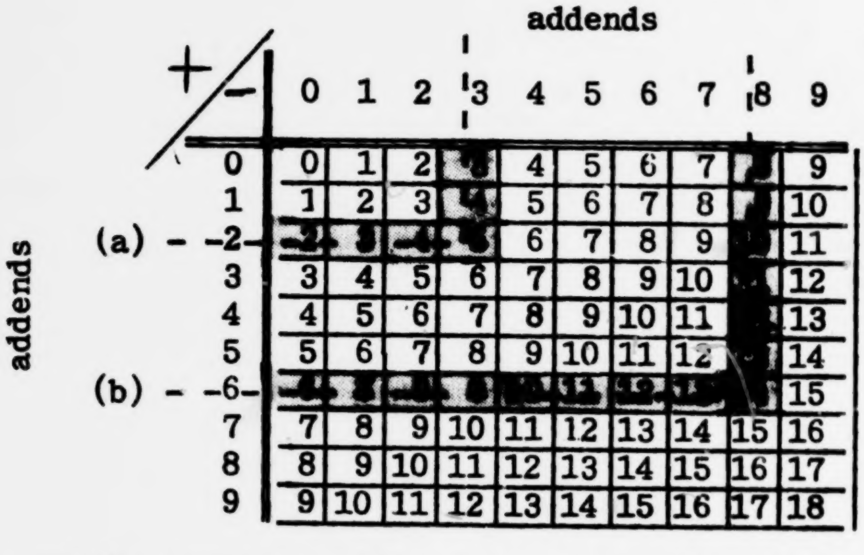
$14 + 6 = 20$ $\qquad\qquad$ $20 - 6 = \boxed{14}$

After examining several subtraction examples on the chalkboard with the pupils, have them practice the following at their desks, using the number line. (Write these on the chalkboard.)*

(a)	(b)
$14 + 9 = \boxed{23}$	$32 - 24 = \boxed{8}$
$23 - 14 = \boxed{9}$	$19 - \boxed{9} = 10$
$33 - \boxed{11} = 22$	$\boxed{19} - 15 = 4$

The pupils will probably show their work in a variety of ways for each example shown above. Discuss the work thoroughly. Let each pupil with a different illustration explain his work.

Another illustration for showing the relationship between addition and subtraction is through the use of the Addition-Subtraction Chart. (See illustration.)


addends
addends
(a)
(b)

+/−	0	1	2	3	4	5	6	7	8	9
0	0	1	2	[illegible]	4	5	6	7	[illegible]	9
1	1	2	3	4	5	6	7	8	[illegible]	10
2	2	[illegible]	4	[illegible]	6	7	8	9	[illegible]	11
3	3	4	5	6	7	8	9	10	[illegible]	12
4	4	5	6	7	8	9	10	11	[illegible]	13
5	5	6	7	8	9	10	11	12	[illegible]	14
6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	11	12	[illegible]	[illegible]	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

The numerals at the top and on the left of the chart are the addends.

---

*The discovery of more difficult subtraction combinations on the number line certainly does not mean that the pupils will retain them after seeing the illustration once. Thus, practice will be necessary, although the type and amount of practice needed will differ for individual members of the group. The use of the number line will enable students to "reason" out an answer when the number combination is unknown or forgotten. Moreover, this experience will help them in working with fractions, decimals, and other numeration systems.

In the illustration, example (a) can be interpreted as follows:

Addition	Subtraction
2 + 3 = 5	5 - 3 = 2
3 + 2 = 5	5 - 2 = 3

In example (b) we have:

6 + 8 = 14	14 - 8 = 6
8 + 6 = 14	14 - 6 = 8

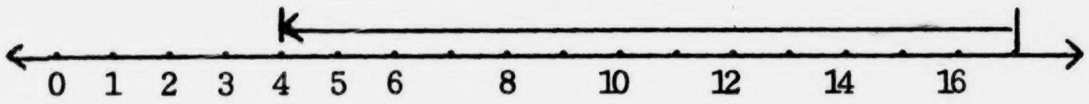
The point of intersection for addition, example (a), is 5 and the point of intersection for example (b), is 14. To subtract, we begin at the point of intersection, subtracting one addend and finding the missing addend.

Pupil Exercises. Provide each pupil with worksheet No. 13.

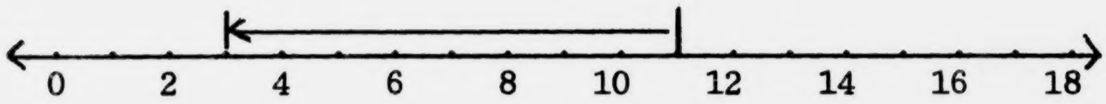
Make sure that each pupil has a ruler or a straightedge for this activity. With the less able group, it may be necessary for the teacher to read problems 6, 7, and 8 with the pupils.

Interim Mathematics Unit - Grade 6 Worksheet No. 13
(San Diego City Schools (1-65)

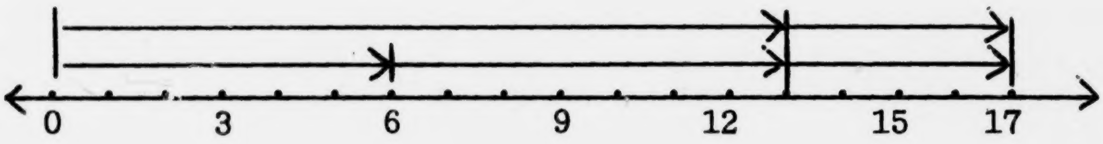
WRITE THE NUMBER SENTENCE FOR EACH ILLUSTRATION WHICH IS SHOWN ON THE NUMBER LINE. USE THE BLANK SPACE AT THE RIGHT OF EACH ILLUSTRATION.

1.


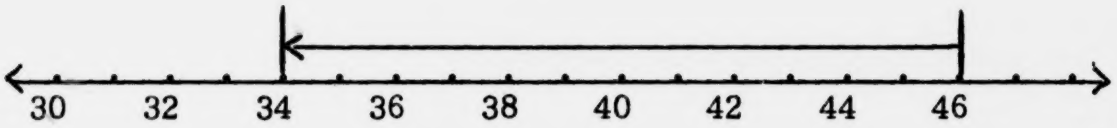
17 - 13 = 4

2.


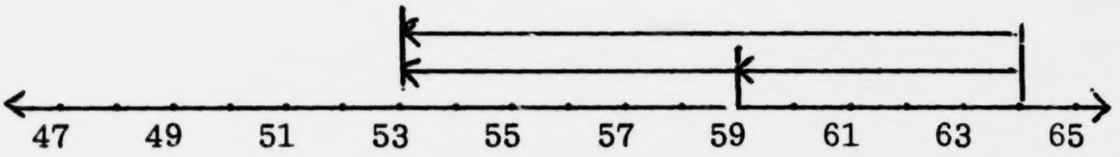
11 - 8 = 3

3.


(6 + 7) + 4 = 17

4.


46 - 12 = 24

5.


(64 - 5) - 6 = 53

SOLVE EACH PROBLEM. USE THE NUMBER SENTENCE FORM.*

6. The games' leaders and the cafeteria helpers from Lee Elementary School were rewarded for their work with a free trip to the San Diego Zoo. Forty-five boys and twenty-nine girls made the trip. How many children went on the free trip?

45 + 29 = 74

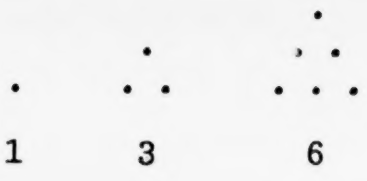
7. At the zoo, two men were selling balloons. Each had 50 balloons to sell. By late afternoon, one man had only 8 left and the other had 12 left. How many balloons had they sold?

100 - (8 + 12) = 100 - 20 = 80

8. The children traveled to the zoo by bus, a distance of 12 miles. At the zoo they traveled another 10 miles. Before returning to school, they took a 47-mile sightseeing tour. What was the total distance traveled that day?

(12 + 10) + 47 = 69

DISCOVER THE PATTERN:

9.


10, 15, 21, 28, 36, 45, 55, 66

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*Number sentences will vary.

3. Commutative Principle of Addition

Objective. To help pupils understand that in the operation of addition the order of the addends may be changed without changing the sum. For example, 3 + 5 = 5 + 3 = 8.

Background Information for the Teacher. The operation of addition has many properties (characteristics). One of the basic properties is the commutative principle. This principle states that the sum of two numbers is the same regardless of the order in which the numbers are added. This means that 12 + 3 = 3 + 12 whether or not we know the sum to be 15. Understanding this principle greatly facilitates learning the addition combinations. When the pupils discover, for example, that 7 + 9 = 16, by applying the commutative principle of addition, they have also learned that 9 + 7 = 16.

This principle may also be stated as:

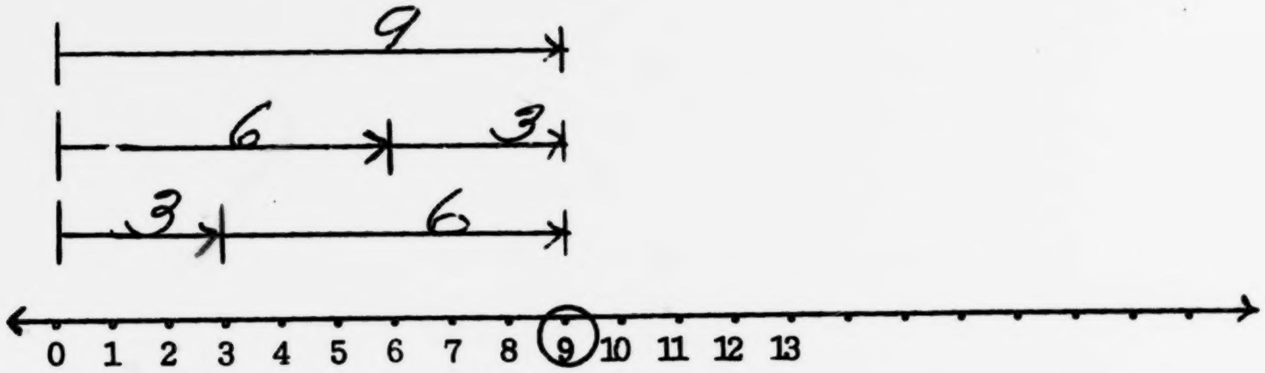
> "if a and b represent any whole numbers, then
> a + b = b + a."

Remember that a principle must hold true for all numbers, that is, the statement a + b = b + a is true for all pairs of whole numbers, a and b.

TEACHING PROCEDURE

Exploration. Non-numerical and numerical examples of combining things should be introduced by using concrete objects such as erasers and rulers to indicate that two sets (groups) of objects may be put together in two different ways. Emphasize that the order of putting sets of objects together does not affect the number of objects.

Next, use the number line to illustrate the following number sentence example: 3 + 6 = 6 + 3 = 9



Point out to the pupils that in using the number line we can reach agreement that the number line illustration is useful in thinking of both 3 + 6 = 9 and 6 + 3 = 9. If this is established early in the use of this unit, pupils will have a graphic representation of the commutative principle of addition. (The use of the phrase commutative principle with pupils of this grade level is optional.)

Use other examples to illustrate the commutative principle of addition, that is,

(a) 14 + 12 = 12 + 14

(b) 24 + 10 = 10 + 24

(c) 36 + 14 = 14 + 36

Next, indicate to the pupils that the order of addends in arithmetic as compared to the order of letters in the English language and the order of events or ways of doing things sometimes vary. For example, ask the following questions:

1. Pretend you have a dime and a quarter. You are paying for a 35¢ book with a quarter and a dime. Would the order of giving the coins to the clerk make a difference in what you paid him? No

2. Use the letters "o" and "n" to make two words. What are the two words? on and no. Did the order of the letters change the result? yes.

3. Is the order important when you:

   (a) Season your food with salt and pepper? No
   (b) Put your socks and shoes on? yes
   (c) Roast weiners and eat them? yes
   (d) Fix a flat and ride your bicycle? yes

4. What can we say about the order of addends in arithmetic as compared to the order of letters in the English language and the order of events or ways of doing things? In arithmetic changing the order of addends does not affect the sum; in the English language and in the order of events, the results are not always the same.

5. What can we say about each of the following? (Write these on the chalkboard.)

   (a) 40 + 50 = 50 + 40
   (b) 69 + 8 = 8 + 69
   (c) 687 + 596 = 596 + 687
   (d) 1,000,000 + 1 = 1 + 1,000, 000

   The order of addends is changed; however, the sum in each case is not affected. Each is a true statement.

The important outcome of these examples is to show that the order of adding two whole numbers does not change the sum.

Encourage pupils to state this principle in their own words. Once they have mastered the meaning of this generalization and can make a statement, inform them that the name for this principle is the commutative principle of addition.

To discover if the commutative principle of addition applies to subtraction, display 12 objects and ask the pupils to "take away" 7 objects. Also, ask why it is possible to get a difference of 5. Next, direct their attention to changing the idea; that is, to see if 12 objects can be "taken away" from 7 objects.

On the chalkboard, write the expression 12 - 7 = 7 - 12 and ask: "Will someone explain why this is not a true statement? Can someone show us this idea through the use of objects?"

Next, encourage the pupils to examine the following illustrations to see if the commutative principle always holds true as it does in addition. Consider these subtraction examples:

(a)	(b)
12 - 7 = 5	63 - 13 = 50
7 - 12 = ☐	13 - 63 = ☐

Ask the following questions:

(1) "In illustration (a), 12 - 7 = 5, if we change the order of the numbers to be subtracted so that we have 7 - 12 = ☐, do we get a whole number for an answer?" No

(2) "Are the answers to 12 - 7 = 5 and 7 - 12 = ☐ the same?" No

(3) "Examine illustration (b). If we change the order of the numbers to be subtracted, do we get a whole number for an answer?" No

(4) "Does the commutative principle of addition always hold true?" Yes

(5) "Is there a commutative principle for subtraction? Defend your answer."*

---

*For those pupils who now immediately want to assume that there exists a principle for subtraction based on the above, ask, "Does verification for one or even several examples establish a principle?" A pupil needs to recognize that a principle is a generalization about numbers that must hold true in every instance. Since we can find instances where the generalization is false, showing it is not true in each instance, there is no commutative principle for subtraction. (Use this same explanation when discussing the operation of division, also.)

Next, review with the pupils the Addition-Subtraction Chart on page 76 which was introduced earlier. When the pupils become familiar with this chart in locating patterns, introduce them to "magic grids." Shown below is a grid with 15, 6, 8, and 18. Encourage pupils to find the pattern by adding two numbers whose numerals appear in the interior region of the grid. Have them write each sum in two places, thus showing that the sum is the same regardless of the order in which the numbers are added. Emphasize that in the "magic grid" we may add up and down, across in either direction, and on the diagonal.

33	23	24	14
21	15	6	21
26	8	18	26
14	23	24	33

This grid shows that numbers can be paired and added regardless of order. For example, 15 + 6 = 6 + 15; 8 + 18 = 18 + 8; 15 + 18 = 18 + 15; etc.

Have the pupils study the magic grid and ask, "How many addition combinations are shown? 12 Is a subtraction sentence related to each addition sentence?" Yes; for example, if 6 + 18 = 24, then 24 - 18 = 6 and 24 - 6 = 18.

On the chalkboard construct a magic grid with only the numeral 6 shown in the interior of the grid and the numerals 15, 14, and 13 shown in the exterior.

16	15	15	14
13	7	6	13
17	8	9	17
14	15	15	16

Ask, "Can we find what the other interior numerals are if the exterior numerals represent the sum?" Yes; for example, 15 - 6 = 9; 14 - 6 = 8; 13 - 6 = 7; etc.

Encourage pupils to complete the "magic grid" by adding any two numbers whose numerals appear in the interior region of the grid. Have them write each sum in two places in the exterior region.

Construct the following grids on the chalkboard and have the pupils put in the missing numerals:

11	44	23	56
22	5	17	22
45	39	6	45
56	44	23	11

16	21	32	37
30	7	23	30
23	14	9	23
37	21	32	16

Pupil Exercises. Provide each pupil with worksheet No. 14.

Limit the explanation of this worksheet with the more able pupils. Ask them to interpret the directions. If their interpretation is correct, let them proceed independently. With the less able group, offer as much explanation as is necessary. Make example 9 optional for the less able group or work the example cooperatively at the chalkboard with this group.

Interim Mathematics Unit - Grade 6 Worksheet No. 14
San Diego City Schools (1-65)

USE THE COMMUTATIVE PRINCIPLE OF ADDITION TO COMPLETE EACH NUMBER SENTENCE.

		If	Then
1.	24 + 17 = 17 + 24	24 + 17 = 41	17 + 24 = 41
2.	36 + 27 = 27 + 36	36 + 27 = 63	27 + 36 = 63
3.	85 + 15 = 15 + 85	85 + 15 = 100	15 + 85 = 100
4.	53 + 18 = 18 + 53	53 + 18 = 71	18 + 53 = 71

DRAW A CIRCLE AROUND THE LETTER OF EACH EXAMPLE WHICH SHOWS THE COMMUTATIVE PRINCIPLE OF ADDITION.

5. [✓ (a)] 43 + 13 = 13 + 43
(b) 65 + 0 = 0 + 56
[✓ (c)] 34 + 37 = 37 + 34
[✓ (d)] (5 + 8) + 27 = 27 + (5 + 8)
[✓ (e)] 18 + (0 + 8) = (0 + 8) + 18
[✓ (f)] 17 + 0 = 0 + 17

WRITE TWO NUMBER SENTENCES FOR EACH OF THE FOLLOWING:

6. Forty-five pupils from Balboa Elementary School went to Knott's Berry Farm by commercial bus. Eighteen went in several family cars. How many pupils went altogether?

45 + 18 = 63 or 18 + 45 = 63

7. Twenty-four pupils went to Marineland in 5 different cars. Fifty-five went by chartered bus. How many pupils made the trip?

24 + 55 = 79 or 55 + 24 = 79

WRITE A "word problem" FOR THE FOLLOWING NUMBER SENTENCE:* 64 + 33 = 97

8. ______________________________

COMPLETE THE FOLLOWING MAGIC GRIDS: (Find the sums.)

9.

51	68	43	60
62	35	27	62
49	33	16	49
60	68	43	51

64	56	83	75
73	27	46	73
66	29	37	66
75	56	83	64

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*Problems will vary.

4. Associative Principle of Addition

Objectives. (1) To help pupils understand that when adding numbers, various groupings are permissible. (2) To help pupils understand that addition is a binary operation; namely, just two numbers can be added in one addition operation.

Background Information for the Teacher. When adding three or more numbers, the addends may be grouped or paired together in more than one way without affecting the sum. For example, in adding 2 + 3 + 4, we may group the addends as (2 + 3) + 4 or as 2 + (3 + 4) and in either case will get the sum of 9. In the first case, we may say 5 + 4 = 9, and in the second, 2 + 7 = 9. This principle is named the associative principle of addition.

Since addition is a binary operation, an equation such as $2 + 3 + 4 = \square$ does not tell us what numbers to group or which addition operation to perform first. However, because of the associative principle of addition, we may group the addends as $(2 + 3) + 4 = \square$ and $2 + (3 + 4) = \square$ without affecting the sum. In both cases, we are really dealing with two numbers at a time.

Parentheses are used to indicate which operation is to be performed first. We shall use it for the sole purpose of indicating (a) what numbers are grouped together and (b) what operation is to be performed first.

When adding two numbers whose sum is greater than 9, it is often convenient to rename the numbers. For instance, notice how different names for the numbers five and ten and how the associative principle of addition are used in the following example:

$9 + 5 = \square$	
$9 + (1 + 4) = \square$	Rename 5 as (1 + 4)
$(9 + 1) + 4 = \square$	Associate addends
$10 + 4 = \square$	Adding 9 + 1
$14 = \square$	Place-value (10 + 4 means 14)

It is possible to always have the numeral or multiple of 10 appear first since the order of the addends does not affect the sum. For example, the addition of $5 + 9 = \square$ can be done in the following ways:

(a)	(b)
$5 + 9 = \square$	$5 + 9 = \square$
$5 + (5 + 4) = \square$	$9 + 5 = \square$
$(5 + 5) + 4 = \square$	$9 + (1 + 4) = \square$
$10 + 4 = \boxed{14}$	$(9 + 1) + 4 = \square$
	$10 + 4 = \boxed{14}$

TEACHING PROCEDURE

Exploration. On the chalkboard write the number sentence 6 + 7 + 5 = ☐. Let the pupils experience for themselves that no matter which grouping they choose, the first step to the solution of the number sentence may be the addition of two of the three numbers.

Since the pupils have been introduced to parentheses earlier in the unit, it might be helpful to review the use of and the need for parentheses.

Next, ask, "Is it possible to add three or more numbers at a time?" No, because addition is a process of thinking about two numbers and getting a third number.

Using parentheses, "Who can show us several examples of grouping the numbers in this number sentence which is written on the chalkboard? Do not change the arrangement of the numerals." Responses should be:

(6+7)+5 = ☐
6+(7+5) = ☐

Now have the pupils examine the following examples which you have written on the chalkboard:

(a) 45 + 15 + 12 = ☐

(b) 33 + 17 + 19 = ☐

(c) 14 + 18 + 12 = ☐

Ask the following questions:

1. "In example (a), what two numbers should be grouped to make addition easier? (45+15) + 12 = ☐

   Would the sum change if we grouped 45 + (15 + 12) = ☐ ?" No

2. "What numbers should we group in example (b)? (33+17)+ 19 = ☐
   If we group 33 + (17 + 19) = ☐, what would happen to the sum?"

   It would be the same.

3. "What grouping would be best in example (c)? 14+(18+12)= ☐
   Is it true that (14 + 18) + 12 is equal to 14 + (18 + 12)?"

   Yes, because the sum is the same.
   Yes, because of the associative principle of addition.

Point out to the pupils that in each example the arrangement of the numerals was not changed; only parentheses were used to group two numbers.

For practice, write several examples on the chalkboard and have the pupils copy and solve each example. Have them use parentheses and show two different groupings. In grouping each example in two ways, have them observe what happens to the sum. Indicate that the arrangement of the numerals must not be changed. The teacher may want to use the following examples:

(a) 14 + 16 + 22 = □
(b) 37 + 13 + 10 = □
(c) 0 + 27 + 13 = □
(d) 16 + 19 + 1 = □
(e) 23 + 22 + 8 = □
(f) 35 + 15 + 0 = □

A sample of each pupil's work should look similar to the following:

$(14+16)+22=\square$
$30+22=\underline{52}$

$14+(16+22)=\square$
$14+38=\underline{52}$

When examples similar to the above have been completed, the teacher may ask leading questions which might guide the pupils to make a generalization. For example, ask:

(1) "Why are parentheses useful in arithmetic?"
(a) They help to group two numbers.
(b) They indicate which operation (which group) we must work with first.

(2) "Was the arrangement of numerals changed?" No.

(3) "What can we say about the order of grouping numbers in addition?"
The order of grouping does not affect the sum.

Inform the pupils that they have been working with a principle which is called the associative principle of addition.

If some pupils have difficulty understanding this principle, the teacher may find the number line helpful in graphically demonstrating that "numbers to be added may be grouped in different ways without affecting the sum."

Pupil Exercises. Provide each pupil with worksheet No. 15.

Limit the explanation of this worksheet with the more able pupils. Ask them to interpret the directions. If their interpretation is correct, let them proceed independently. With the less able group, offer as much explanation as is necessary.

Worksheet No. 15

USE PARENTHESES TO SHOW THAT THE ORDER IN WHICH NUMBERS ARE GROUPED DOES NOT AFFECT THE SUM. FOR EXAMPLE:

(42 + 18) + 15 = 42 + (18 + 15)
60 + 15 = 42 + 33
75 = 75

(a)

1. (33 + 7) + 35 = 33 + (7 + 35)
40 + 35 = 33 + 42
75 = 75

(b)

(36 + 14) + 29 = 36 + (14 + 29)
50 + 29 = 36 + 43
79 = 79

2. (63 + 0) + 27 = 63 + (0 + 27)
63 + 27 = 63 + 27
90 = 90

(28 + 35) + 25 = 28 + (35 + 25)
63 + 25 = 28 + 60
88 = 88

WRITE THE MISSING NUMERAL FOR EACH ○ WHICH WILL MAKE THE NUMBER SENTENCE TRUE.

(a)

3. 45 + (12) = 12 + 45
4. (38) + 47 = 47 + 38
5. (17 + 33) + 28 = 17 + ((33) + 28)

(b)

65 + (22) = 22 + 65
(18 + 16) + 29 = (18) + (16 + 29)
33 + (18 + 39) = (33 + (18)) + 39

SOLVE THE EXAMPLES. (Hint: Think of the addition principle which you have learned.)

(a)

6. 63 + 59 = 63 + 40 + 19
7. 48 + 26 = 40 + 8 + 26
8. 40 + 65 = 65 + 10 + 30
9. 33 + 76 = 33 + 24 + 52

(b)

85 + 24 = 85 + 20 + 4
47 + 28 = 30 + 17 + 28
63 + 27 = 25 + 38 + 27
99 + 19 = 49 + 50 + 19

DRAW A CIRCLE AROUND THE LETTER OF EACH EXAMPLE WHICH IS TRUE.

10. [✓ (a)] (25 + 15) < 45
[✓ (b)] (19 + 11) < 35
(c) (36 + 19) < 54
[✓ (d)] 64 < (19 + 57)
[✓ (e)] (8 + 9) < (9 + 10)
[✓ (f)] 85 < (70 + 16)

5. Extending Addition and Subtraction

Objectives. (1) To extend the regrouping concept to include 10 tens as a hundred and 10 hundreds as a thousand. (2) To promote a better understanding of the conventional algorithms of addition and subtraction.

Background Information for the Teacher. It is important for pupils to understand that the idea of "different names for the same number" is basic for understanding and learning the operations of addition and subtraction.

Consider the following example which should make clear (1) the need to learn how to express numbers in different forms and (2) the useful application of this skill in addition and subtraction:

> "Find the number that must be added to 376 so that 523 will be the sum." If the sentence is written as $376 + \square = 523$, immediate recall of the subtraction combinations is unlikely. Hence, it is helpful to express 523 and 376 in expanded form, that is,

$$\begin{array}{rcl|cl|cl|cl}
 & 523 & = & 500 + 20 + 3 & = & 500 + 10 + 13 & = & 400 + 110 + 13 \\
- & \underline{376} & = & \underline{300 + 70 + 6} & = & \underline{300 + 70 + 6} & = & \underline{-(300 + 70 + 6)} \\
 & \square & & & & & = & 100 + 40 + 7 \\
 & & & & & & = & 147
\end{array}$$

The primary issue is that of recognizing the need for more ones in the ones' place and more tens in the tens' place.

In the following example there is less difficulty. We simply name the addends to exhibit the number of ones, the number of tens, etc. Then the ones, tens, hundreds, etc. are added. For example:

$$\begin{array}{rcll}
249 & = & 200 + 40 + 9 & \\
+\ 676 & = & \underline{600 + 70 + 6} & \\
 & = & 800 + 110 + 15 & = (800 + 110) + 15 \\
 & & & = 910 + 15 \\
 & & & = 925
\end{array}$$

Here the numeral (800 + 110 + 15) is expressed as the numeral by applying properties of our system of numeration and the commutative principle of addition and the associative principle of addition.

TEACHING PROCEDURE

Exploration. On the chalkboard write the following problem:

"Cindy put 89 animal pictures in her social studies album on Saturday and 71 pictures on Sunday. How many pictures did she mount in the two days?"

Next, write the problem in expanded form for analysis such as:

89 = ☐ tens, ☐ ones

+ 71 = ☐ tens, ☐ ones

___ tens, ___ ones

Ask the following questions: ✲

(1) "How many ones do we get when we add?" 10

(2) "Can we regroup 10 ones as 1 ten, 0 ones?" Yes

(3) "If we add 1 ten to 8 tens + 7 tens, how many tens will we have?" 16

(4) "Is it true that 89 + 71 is 16 tens, 0 ones?" Yes

(5) How many hundreds, tens, and ones in 16 tens and 0 ones?" 1 hundred, 6 tens, 0 ones

To show regrouping of 10 tens as a hundred, write the following example on the chalkboard and guide the pupils in analyzing the process through questions such as: ✲

264 = ___ hundreds, ___ tens, ___ ones

+ 182 = ___ hundred, ___ tens, ___ ones

= ___ hundreds, ___ tens, ___ ones

(1) "Do we need to regroup the ones?" No

(2) "How can we regroup the sum of 6 tens + 8 tens?" 1 hundred + 4 tens

(3) "When we add the 1 hundred to the other hundreds, how many hundreds, tens, and ones will we have?" 4 hundreds, 4 tens, 6 ones

(4) "What is the sum of 264 + 182 = ☐ ?" 446

✲ As the pupils respond to each question, write the numeral in each appropriate frame.

Using the same pattern of analysis, the teacher may wish to use the following example to illustrate regrouping in the ones', tens', and hundreds' place:

818 = 8 hundreds, 1 ten, 8 ones

+ 383 = 3 hundreds, 8 tens, 3 ones

= 11 hundreds, 9 tens, 11 ones

= 11 hundreds, 10 tens, 1 one

= 12 hundreds, 0 tens, 1 one

= 1 thousand, 2 hundreds, 0 tens, 1 one

= 1000 + 200 + 0 + 1

= 1,201

The example shown above presents a detailed analysis of regrouping. Time should be taken to work the example cooperatively with the pupils. When the concept is well understood, write several examples on the chalkboard for the pupils to do at their desks. Examples such as these will be appropriate:

895　　677　　723
+ 567　　+ 323　　+ 489

Regrouping tens and hundreds to subtract is an extension of the developmental work initiated earlier in this chapter. You may wish to introduce this regrouping by writing the following problem on the chalkboard:

> "Bob lives near a lake where men from the Wildlife Service catch wild ducks. The men attach bands to the ducks in order to study their living habits. One day 328 ducks were banded. The next day only 134 ducks were banded. How many more ducks were banded on the first day than on the second day?"

Through carefully planned, guided discussion questions, have the pupils assist in stating the problem, using the long form, that is, *

328 = ____ hundreds, ____ tens, ____ ones

− 134 = ____ hundred, ____ tens, ____ ones

____ hundreds, ____ tens, ____ ones

Ask, "How can we regroup 328 so that we can subtract the tens? 2 hundred 12 tens, 8 ones. Did regrouping take place in the ones' place? No If we write the problem in number sentence form such as 328 − 134 = ☐, what is the missing numeral (difference)?" 194

To illustrate regrouping both the tens and the hundreds, use the following example: *

932 = ___ hundreds, ___ tens, ___ ones
- 576 = ___ hundreds, ___ tens, ___ ones
= ___ hundreds, ___ tens, ___ ones

Ask, "How can we regroup 932 before we can subtract?" First, change 932 to 9 hundreds, 2 tens, 12 ones; this will make subtraction in the ones' place possible. Next, change the 9 hundreds, 2 tens to 8 hundreds, 12 tens; this makes it possible to complete the subtraction process.

After completing the subtraction, as, "What numeral is another name for 3 hundreds, 5 tens, 6 ones?" 356

For individual practice, have pupils work several of the following examples at their desks:

(a)	(b)	(c)	(d)
819	829	589	904
- 453	- 547	- 390	- 576

Pupil Exercises. Provide each pupil with worksheet No. 16.

Encourage pupils to see relationships with respect to the operation of addition and subtraction. The less able group may need more introductory work before they can do these examples independently.

Examples 7(a) and (b) may be done cooperatively with the less able pupils by duplicating the grids on the chalkboard.

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* As the pupils respond to your questions, write the numerals in the appropriate blank spaces.

FIND THE MISSING NUMERAL FOR EACH BLANK SPACE IN THE FOLLOWING EXAMPLES:

Addition

1. 127 = 1 hundred, 2 tens, 7 ones
+ 238 = 2 hundreds, 3 tens, 8 ones
= 3 hundreds, 5 tens, 15 ones
= 365

Subtraction

365 = 3 hundreds, 5 tens, 15 ones
− 238 = − 2 hundreds, 3 tens, 8 ones
= 1 hundreds, 2 tens, 7 ones
= 127

2. 485 = 400 + 80 + 5
+ 737 = 700 + 30 + 7
= 1100 + 110 + 12
= 1,222

1,222 = 1,100 + 110 + 12
− 737 = − 700 + 30 + 7
= 400 + 80 + 5
= 485

3. What discovery did you make about examples 1 and 2 above?* Addition and subtraction are related. In addition we want to find the sum; in subtraction we want to find the missing addend.

COMPLETE THE FOLLOWING EXAMPLES:

(a)

4. 45 + 38 = 83
5. 65 + 25 = 90
6. 128 + 222 = 350

(b)

83 − 45 = 38
90 − 25 = 65
350 − 222 = 128

FIND THE MISSING SUMS AND MISSING ADDENDS FOR THE FOLLOWING GRIDS:

(a)

52	67	49	64
93	48	45	93
23	19	4	23
64	67	49	52

(b)

158	162	96	100
128	95	33	128
130	67	63	130
100	162	96	158

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*Accept any answer which will show that subtraction and addition are related.

B. Multiplication and Division

Multiplication of Whole Numbers

Objective. To review and extend the meaning of multiplication, using sets, arrays, and number sentences.

Background Information for the Teacher. Multiplication of whole numbers can be defined in terms of sets. For example, the illustration below represents an array* of 3 rows and 4 columns. The number associated with this array is 3 x 4. Since 3 x 4 is not a standard form for expressing a counting number, we can set up a one-to-one correspondence between the members of the set (illustration) and the members of the set of counting numbers. Thus, the number property of the 3 x 4 array is 12. We can then name

4 members
(columns)

* * * *

3 sets * * * *
(rows)

* * * *

the number of the illustration as 3 x 4 or 12.

In 3 x 4 = 12, the 3 and 4 are factors; the 12 is called the product of 3 x 4. Thus we have:

Factor		Factor		Product
3	x	4	=	12

It is often useful to think of multiplication in terms of repeated addition. For example,

3 x 5 is equal to 5 + 5 + 5 or three 5's

6 x 2 is equal to 2 + 2 + 2 + 2 + 2 + 2 or six 2's

The repeated addition approach to multiplication has limitations. Using repeated addition to solve the problem "how many quarts are there in 172 gallons" is too time consuming, since 4 would be used as an addend 172 times. Thus, the operation of multiplication is necessary which provides the element of economy.

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*See Glossary, page 179.

TEACHING PROCEDURE

Exploration. Introduce the pupils to the word array and show several illustrations which may represent an array. Empty egg cartons may serve as good representations. Proceed from a physical model to the graphic representation, for example:

Array Examples	Definition	Meaning	Number Sentence
* * * * * * * * * * * *	3 sets with 4 members in each set	three 4's	3 x 4 = 12
* * * * * * * * * * * *	2 sets with 6 members in each set	two 6's	2 x 6 = 12

Next, represent other arrays on the chalkboard and have pupils respond orally, identifying each example in several ways. Example (a) will serve as a model:

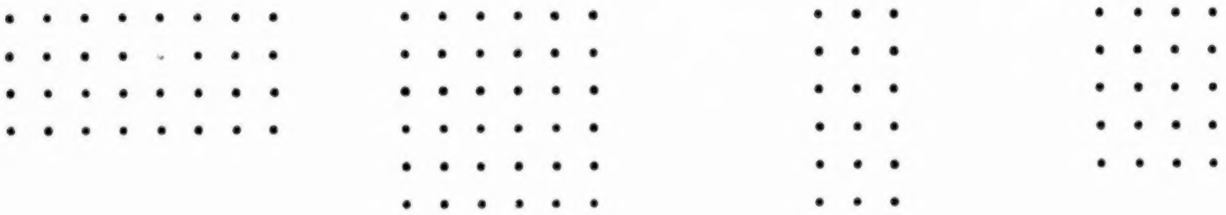
(a)	(b)	(c)
* * * * * * * * * * * * * * *	. . . .	△ △ △ △ △ △ △ △ △ △ △ △
3 × 5 = 15	4 X 6 = 24	4 X 3 = 12
three 5's = 15	four six's = 24	four 3's = 12
5 + 5 + 5 = 15	6+6+6+6 = 24	3+3+3+3 = 12
3 + 3 + 3 + 3 + 3 = 15		4 + 4 + 4 = 12

Write the following examples on the chalkboard and have the pupils assist in writing the sum and a multiplication sentence for each:

(a)	(b)	(c)	(d)
10	12		
10	12	98	109
10	12	98	109
10	12	98	109
+ 10	+ 12	+ 98	+ 109
50	60	392	436
5 x 10 = 50	5 x 12 = 60	4 x 98 = 392	4 x 109 = 436

Next, have pupils illustrate arrays for each of the following and find the missing numeral for each. Write these on the chalkboard:*

(a)	(b)	(c)	(d)
$4 \times 8 = \boxed{32}$	$6 \times 6 = \boxed{36}$	$\boxed{6} \times 3 = 18$	$5 \times \boxed{4} = 20$



On the chalkboard, write the following number sentences and have the pupils recite a possible number story. For example:

	Number Sentence	Story
(a)	$3 \times 4 = 12$	Bill and Bob found 3 birds' nests in the canyon near Linda Vista. In each nest there were 4 eggs.
(b)	$5 \times 3 = 15$	Alice gathered 3 maple leaves, 3 apple leaves, 3 willow leaves, 3 oak leaves, and 3 cottonwood leaves.

Pupil Exercises. Provide each pupil with worksheet No. 17.

Discuss the worksheet before independent work is assigned. With the less able group, offer as much help as you feel is needed. Problems 9 and 10 may be done cooperatively with the group.

Number sentences in examples 9 and 10 will vary.

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* On the chalkboard x's or small o's should be used since they are easily distinguished from a dot.

FIND THE SUM, THEN WRITE A MULTIPLICATION SENTENCE FOR EACH EXAMPLE:

1. 63 + 63 + 63 + 63 + 63 = 315   5 × 63 = 315

2. 114 + 114 + 114 + 114 + 114 + 114 = 684   6 × 114 = 684

3. 225 + 225 + 225 + 225 + 225 = 1,125   5 × 225 = 1,125

4. 387 + 387 + 387 + 387 + 387 + 387 = 2,322   6 × 387 = 2,322

DRAW AN ARRAY FOR EACH MULTIPLICATION SENTENCE, THEN FIND THE PRODUCT.

	(a)	(b)	(c)
5.	6 × 4 = 24	5 × 7 = 35	4 × 8 = 32
	XXXX XXXX XXXX XXXX XXXX XXXX	XXXXXXX XXXXXXX XXXXXXX XXXXXXX XXXXXXX	XXXXXXXX XXXXXXXX XXXXXXXX XXXXXXXX

FIND THE PRODUCT FOR EACH EXAMPLE:

	(a)	(b)	(c)
6.	284 × 12 3,408	196 × 17 3,332	284 × 36 10,224

7. 43 × 9 = 387   123 × 7 = 861   283 × 4 = 1,132

8. 321 × 6 = 1,926   315 × 8 = 2,520   299 × 10 = 2,990

9. Mark and Greg kept a record of Ford station wagons, 4-door cars, and 2-door cars while on their way to school. They saw 12 of each kind of car. How many different cars did they record altogether?

3 × 12 = □; □ = 36

10. In the Paradise Hills Elementary School auditorium there are 35 rows of chairs with 50 chairs in each row. How many chairs are there altogether?

35 × 50 = □; □ = 1,750

2. Commutative Principle of Multiplication

Objective. To help pupils understand and use the commutative principle of multiplication which is exemplified by examples such as: 5 × 7 = 7 × 5, 6 × 4 = 4 × 6, etc. These examples illustrate that the order of the factors does not affect the product.

Background Information for the Teacher. Multiplication and addition have certain properties in common: (a) Both are binary operations, that is, only two numbers can be operated upon at one time. (b) Both are commutative. An example of the commutative principle of addition is 3 + 8 = 8 + 3, and an example of the commutative principle of multiplication is 5 × 4 = 4 × 5. In the addition example the sum did not change; in the multiplication example, the product did not change.

Multiplication of two numbers may be thought of in various ways. We may think of them as the number to be multiplied (or repeatedly added) and the number of times it is to be multiplied (or added).

TEACHING PROCEDURE

Exploration. Review with the pupils the commutative principle of addition. Next, write the following examples on the chalkboard and ask the pupils to analyze them in terms of their effect on the product.

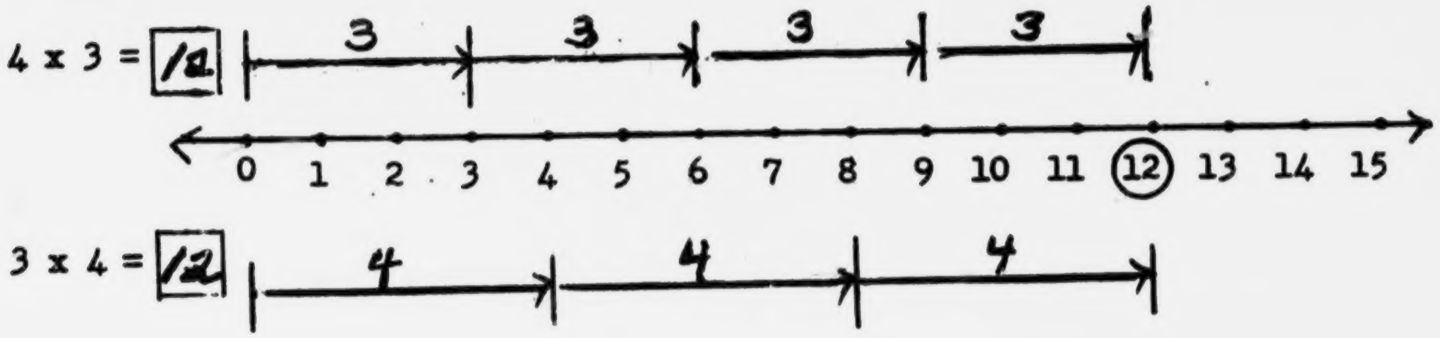
(a)	(b)
12 × 5 = 60	5 × 12 = 60
15 × 6 = 90	6 × 15 = 90
4 × 16 = 64	16 × 4 = 64
17 × 5 = 85	5 × 17 = 85

Ask the pupils, "In each corresponding example, what can we say about the product? The product in each example is the same.

How does changing the order of the factors affect the product? The product remains the same. What statement can we make regarding the change in the order of factors which will always hold true?"

In multiplication changing the order of the factors does not affect the product.

Using the same number line ask someone to come to the chalkboard to show that 4 x 3 = 3 x 4. The illustration might look like this:



Ask the following questions:

(1) "What happens when we change 4 × 3 to 3 × 4 on the number line?" The product is the same but the steps are different.

(2) "What does 3 × 4 (product expression) tell us?" That there are 3 sets, each having 4 members.

(3) "Does the product expression 4 × 3 differ from the product expression 3 × 4?" Yes. 4X3 shows that there are 4 sets with 3 members in each set; 3X4 shows that there are 3 sets, each having 4 members.

(4) "Does the order of the factors affect the product?" No.

Use other examples such as 5 × 4, 9 × 3, 10 × 2, etc. and ask questions similar to those listed above.

While some of the pupils work at the chalkboard, have others work at their desks. Pupils should be encouraged to construct their own number lines.

3. Associative Principle of Multiplication

Objectives. (1) To help pupils understand that in multiplication various groupings are permissible. (2) To help pupils understand that multiplication is a binary operation; namely, just two numbers can be multiplied at one time.

Background Information for the Teacher. In multiplying three or more numbers, the factors may be grouped or paired together in more than one way without affecting the product. For example, in multiplying 3 x 4 x 5, we may group the factors as (3 x 4) x 5 or as 3 x (4 x 5). In either case, the product is 60. This property is named the associative principle of multiplication.

In multiplication it is sometimes necessary to find products involving more than two factors, for example, 4 x 2 x 3 = ☐ . Since the operation of multiplication is associative, the number sentence can be solved.

As in addition, parentheses are used to state which operation is to be performed first. We shall also use it for indicating what numbers belong together.

Because of the associative principle of multiplication and the commutative principle of multiplication, we can group the factors and multiply them in any order. In the number sentence 9 × 2 × 5 = ☐, we can multiply from left to right or from right to left. For example:

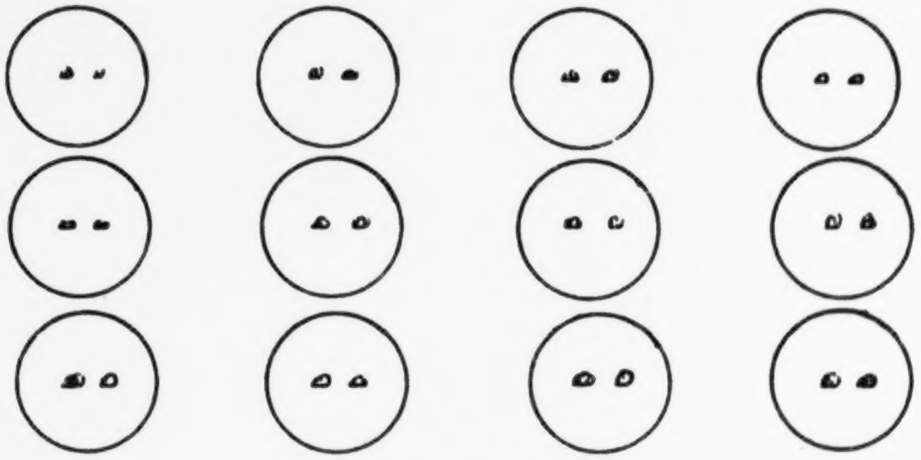
$$(9 \times 2) \times 5 = 18 \times 5 = 90$$

$$9 \times (2 \times 5) \overset{\text{or}}{=} 9 \times 10 = 90$$

$$(9 \times 5) \times 2 \overset{\text{or}}{=} 45 \times 2 = 90$$

TEACHING PROCEDURE

Exploration. To provide pupils with a graphic illustration of the associative principle of multiplication, construct the following example on the chalkboard. This illustration represents 3 sets of buttons with 4 buttons in each set and each button has 2 holes.



Ask the following questions:

(1) "How many sets of buttons do we have? 3. How many are in each set?" 4

(2) "How many holes are in each button?" 2

(3) "Is it possible to illustrate this chart in multiplication sentence form?" Yes

(4) "Who would come to the chalkboard and write the sentence and name each numeral?"

3	x	4	x	2	=	24
sets		members		holes		product

(5) "Show with parentheses which two numbers we are going to multiply first?" (3 x 4) x 2

(6) "What does each numeral in the sentence (3 x 4) x 2 tell us?"
The (3 x 4) tells us the number of members or buttons in the array; the 2 tells us the number of holes in each member.

(7) "What is the product for the multiplication sentence (3 x 4) x 2 = $\boxed{24}$?"

(8) "If we grouped (4 x 2) so that our sentence would look like this: 3 x (4 x 2) = $\boxed{24}$, would the product change?" No.

(9) "What does each numeral in the equation 3 x (4 x 2) = $\boxed{24}$ tell us?

3	x	(4 x 2)	=	$\boxed{24}$
sets		holes in each set		product

(10) "What can we say about the order of factors when we multiply?"
The order does not affect the product.

To reinforce the associative principle of multiplication, write several practice examples on the chalkboard. Have some pupils come to the chalkboard while others work independently at their desks. Direct the pupils to group pairs of factors, without changing the arrangement, to see if the order in which they group the factors will affect the product.

Examples:	(a)	(b)
(1)	4 x 3 x 5 = (4 x 3) x 5	4 x 3 x 5 = 4 x (3 x 5)
	= 12 x 5	= 4 x 15
	= 60	= 60

(a)

(2) 5 × 2 × 4 = (5 × 2) × 4
= 10 × 4
= 40

(b)

5 × 2 × 4 = 5 × (2 × 4)
= 5 × 8
= 40

(a)

(3) 6 × 1 × 3 = (6 × 1) × 3
= 6 × 3
= 18

(b)

6 × 1 × 3 = 6 × (1 × 3)
= 6 × 3
= 18

After sufficient practice with a variety of examples, lead the pupils to generalize that "the order of the factors does not affect the product." Use of the phrase "associative principle of multiplication" is optional with sixth-grade pupils.

Pupil Exercises. Provide each pupil with worksheet No. 18.

Limit the explanation for this worksheet with the more able group. However, with the less able group, offer as much help as is needed, especially with the "word problems." The "Just for Fun" problems may be used for class discussion.

Interim Mathematics Unit - Grade 6
San Diego City Schools (1-65)

Worksheet No. 18

COMPLETE THE FOLLOWING:

1. In addition, the order in which we group the addends does not affect the sum.

2. In multiplication, the order in which we group the factors does not affect the product.

WRITE THE MISSING NUMERAL IN EACH BLANK SPACE:

	(a)	(b)
3.	(4 × 9) × 5 = 36 × 5 = 180	7 × (8 × 6) = 7 × 48 = 336
4.	(9 × 7) × 4 = 63 × 4 = 252	(21 + 63) + 81 = 84 + 81 = 165
5.	(33 + 17) + 35 = 50 + 35 = 85	14 × (5 × 20) = 14 × 100 = 1,400
6.	8 × (12 × 3) = 8 × 36 = 288	(47 × 0) × 4 = 0 × 4 = 0
7.	(63 × 7) × 0 = 441 × 0 = 0	23 × (15 × 4) = 23 × 60 = 1,380
8.	(22 × 6) × 1 = 132 × 1 = 132	(35 × 5) × 5 = 175 × 5 = 875

SOLVE EACH PROBLEM. SHOW THAT GROUPING THE ADDENDS IN ADDITION DOES NOT AFFECT THE SUM; THAT GROUPING THE FACTORS DOES NOT AFFECT THE PRODUCT.

9. A ticket salesman at Belmont Park sold 315 tickets for the racer ride in the morning. In the afternoon he sold 285 tickets. That evening he sold 365 tickets. How many tickets did he sell that day?

(315 + 285) + 365 = 965   315 + (285 + 365) = 965

10. Twelve children bought 5 tickets each for the Ferris wheel ride. Each ticket cost 15¢. How much did the children spend for the ride?

12 × (5 × 15) = $9.00   (12 × 5) × 15 = $9.00

JUST FOR FUN.

11. Which is correct: 23 + 27 is 40, or 23 + 27 are 40?

Neither is correct; the sum is 50

12. If you had one match and had to light an oil lamp, a candle, a stove, and a fireplace, which would you light first?

the match

4. Distributive Principle of Multiplication OVER addition

Objective. To assist pupils in understanding the algorithm of multiplication.

Background Information for the Teacher. The distributive principle* is a fundamental property in the multiplication algorithm. An example of this property is: $3 \times (10 + 7) = (3 \times 10) + (3 \times 7)$. This principle differs from the commutative and associative principles in that it always involves two of the four basic operations. The distributive principle shows the relationship between the operations of multiplication and addition. According to this principle, to find the value of $3 \times (4 + 5)$, we can either (a) add the expression $(4 + 5)$ and then multiply the sum by 3 or (b) multiply both 4 and 5 by 3 and then add the products.

Computation in multiplication can be simplified and the multiplication facts, which have been temporarily forgotten, can be reconstructed if the pupil understands the distributive principle of multiplication. This principle can be illustrated by way of the following:

(a) (b)

$$\begin{array}{rcrcl} 7 & = & 4 + 3 & & \\ \times\ 5 & = & \times \qquad 5 & & \\ \cline{1-1}\cline{3-3} 35 & = & 20 + 15 & = & 35 \end{array}$$

In illustration (a) we have the usual algorithm or conventional model; the product is 35. In illustration (b) we renamed 7 as (4 + 3) and multiplied; the product is the same.

Often in multiplication the work is simplified if a two-digit number is expressed as a sum of tens and ones, such as $5 \times 35 = 5 \times (30 + 5)$.

Since addition is commutative, we can also express $5 \times (30 + 5)$ as $5 \times (5 + 30)$. Observe how the commutative principle of multiplication and the distributive principle are used in the solution for illustration (e):

(c) (d) (e)

$$\begin{array}{r} 35 \\ \times\ 5 \\ \hline 25 \\ 150 \\ \hline 175 \end{array} \qquad \begin{array}{r} 30 + 5 \\ \times \qquad 5 \\ \hline 150 + 25 \end{array} = 175 \qquad \begin{array}{rl} 5 \times 35 & = 5 \times (30 + 5) \\ & = (5 \times 30) + (5 \times 5) \\ & = 150 + 25 \\ & = 175 \end{array}$$

$\left.\begin{array}{l} 5 \times 35 = 5 \times (30 + 5) \\ \quad = (5 \times 30) + (5 \times 5) \end{array}\right]$ distributive principle

The only difference between the vertical and the horizontal form is that in the vertical form, illustration (c), 5 × 5 is done before 5 × 30, illustration (d).

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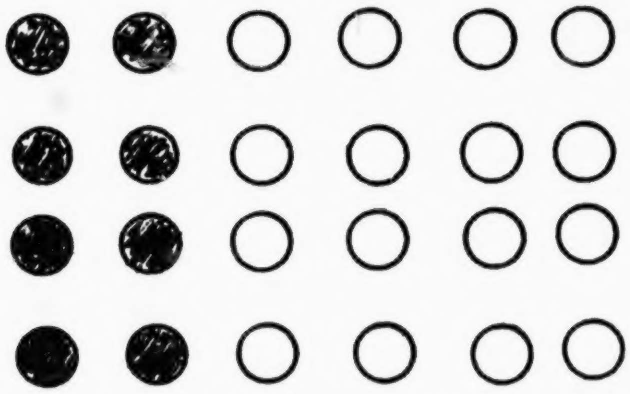
* See GLOSSARY, page 179.

TEACHING PROCEDURE

Exploration. On the chalkboard represent a 4 × 6 array such as the illustration below:



Ask and state the following which may help to make a discovery:

(1) "How many objects are represented in this illustration?" 24

(2) "Write a multiplication sentence for this entire illustration." 4X6=24

(3) "Write a multiplication sentence for those objects which are shaded." 4X2=8

(4) "Write a number sentence for those objects which are not shaded." 4X4=16

(5) "If we add the product of 4 × 2 which is 8 and the product of 4 × 4 which is 16, what will be the sum? 8+16=24 How does the product of 4 × 6 = ☐ compare to the product of 4 × 2 = ☐ plus 4 × 4 = ☐ ?"

The products are the same.

Now, let's examine the following number sentence examples:

(a) 4 × 6 = 24

(b) 4 × 6 = 4 × (2 + 4) = 24

(c) 4 × 6 = (4 x 2) + (4 × 4) = 24

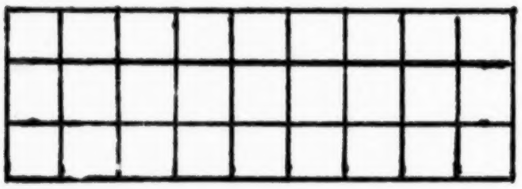
Ask the pupils to make some observations, guiding them to state, for example:

(1) Example (a) has 4 sets and 6 members in each set.

(2) Example (b) renames 6 as (2+4).

(3) Example (c) represents 4 sets of 2 plus 4 sets of 4.

To provide additional understanding of the distributive principle of multiplication over addition, illustrate the following:



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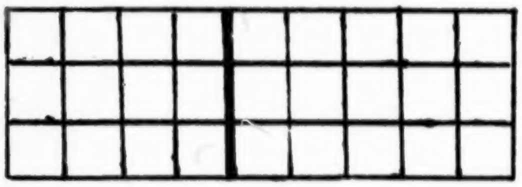
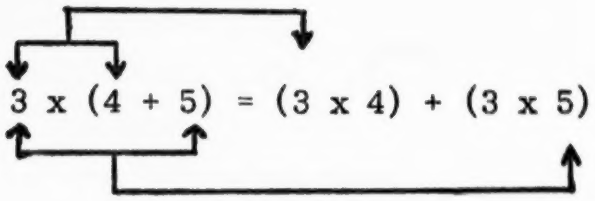


$$3 \times 9 = (3 \times 4) + (3 \times 5)$$

$$= 12 + 15$$

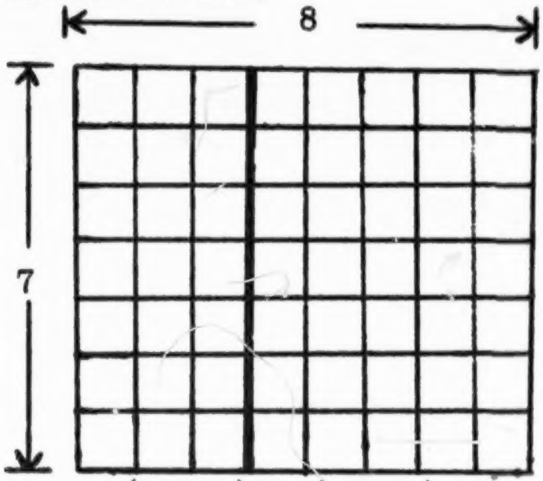
$$27 = 27$$

Help the pupil discover that the distributive principle of multiplication over addition allows us to use the operations of multiplication and addition in the manner shown in the above example. The 3 x 9 array is renamed as a 3 x 4 array plus a 3 x 5 array. The 3 distributes over 4 and over 5, for example:



$$3 \times (4 + 5) = (3 \times 4) + (3 \times 5)$$

Continue by having pupils interpret several more illustrations in the same way. Next, make an array, as illustrated below, from a piece of tagboard that can be folded.



(7 x 3) + (7 x 5)

Show the array to the class and ask, "How many rows and how many columns are in this array? 7 rows and 8 columns
What is the product of this 7 x 8 array?" 56

Now fold the card to show that the 7 × 8 array can be thought of as a 7 × 3 array plus a 7 × 5 array. Ask, "How many members are there in the 7 × 3 array?" 21 (Have this product recorded on the chalkboard.) Next, ask, "How many members are there in the 7 × 5 array?" 35 (Record this product on the chalkboard.) Lastly, ask, "Will the product of (7 × 3) + (7 × 5) be equal to the product of 7 × 8?" yes, because the 8 was renamed as (3+5); the rows and columns did not change.

Continue to illustrate the distributive property of multiplication over addition by folding the same card, 7 × 8 array, in other ways. For example: 7 × (6 + 2); 7 × (7 + 1); and 7 × (4 + 4). Have the related number sentences recorded each time.

For individual practice at their desks and at the chalkboard, write these and similar examples on the chalkboard to clarify the idea that we get the same result whether we add first and then multiply, or whether we multiply first and then add.

(a) 6 × (4 + 2) = (6 × 4) + (6 × 2) = 24 + 12 = 36

(b) 5 × (3 + 6) = (5 × 3) + (5 × 6) = 15 + 30 = 45

Next, have the pupils copy these examples from the chalkboard and put in the missing numerals.

(a) 3 × 8 = (3 × 3) + (3 × 5)
= 9 + 15 = $\boxed{24}$

(b) 4 × 7 = (4 × 6) + (4 × 1)
= 24 + 4 = $\boxed{28}$

(c) 4 × 6 = (4 × 3) + (4 × 3)
= 12 + 12 = $\boxed{24}$

When the pupils become familiar with the distributive principle of multiplication over addition, introduce them to its use when multiplying numbers named by two-place numerals. The emphasis should be on understanding what really happens when we multiply.

For example, illustrate on the chalkboard three ways in which the following example may be performed:

(a)

$$\begin{array}{r} 76 \\ \times\ \ 4 \\ \hline 304 \end{array}$$

(b)

$$\begin{array}{r} 76 \\ \times\ \ 4 \\ \hline 24 \\ \underline{280} \\ 304 \end{array}$$

(c)

$$\begin{aligned} 4 \times 76 &= 4 \times (70 + 6) \\ &= (4 \times 70) + (4 \times 6) \\ &= 280 + 24 \\ &= 304 \end{aligned}$$

Ask, "Which example shows what is really happening when we multiply a two-place numeral by a one-place numeral?" (c), (b).

"Which example appears to conceal the meaning by making it a mechanical operation?" (a)

Provide the pupils with practice work by writing these examples on the chalkboard. Have them copy each example and put in the missing numerals.*

(a) 9 × 21 = (9 × 20) + (9 × 1)

= 180 + 9 or $\boxed{189}$

(b) 5 × 54 = (5 × 50) + (5 × 4)

= 250 + 20 or $\boxed{270}$

(c) 8 × 23 = (8 × 20) + (8 × 3)

= 160 + 24 or $\boxed{184}$

(d) 6 × 45 = (6 × 40) + (6 × 5)

= 240 + 30 or $\boxed{270}$

Multiplying hundreds, tens, and ones (three-place numeral) using the distributive principle of multiplication over addition operates the same way as it does for multiplying a two-place numeral. Point out to the pupils that the result of 3 × 623 may be achieved in several ways. There is the "short-cut" method (a), the "repeated addition" method (b), and "structural methods" (c), (d), and (e). Direct the pupils to examine each of the following:

(a)	(b)	(c)
623	623	623 = 600 + 20 + 3
× 3	623	× 3
1869	+ 623	1800 + 60 + 9 =
	1869	(1800 + 60)+ 9 =
		1860 + 9 = 1,869

---

* All of these examples need not be used; use only those which you feel are necessary to provide the needed practice.

(d)

$$\begin{aligned}3 \times 623 &= 3 \times (600 + 20 + 3)\\ &= (3 \times 600) + (3 \times 20) + (3 \times 9)\\ &= 1800 + 60 + 9\\ &= (1800 + 60) + 9\\ &= 1860 + 9\\ &= 1869\end{aligned}$$

(e)

$$\begin{array}{rl} 623 & \\ \times\ \ \underline{\ \ 3} & \\ 9 & = (9 \times 1)\\ 60 & = (3 \times 20)\\ +\ \underline{1800} & = (3 \times 600)\\ 1869 & \end{array}$$

Now ask, "Which example conceals the meaning of what takes place when we multiply? Example (a) Which example would be 'too long' if we were to show the product of 38 × 432 by repeated addition? Example (b) Which examples have more 'meaning' and actually show what is happening when we multiply?" Examples (c), (d), and (e).

For individual practice, have the pupils copy from the chalkboard each of the following examples and put in the missing numerals:

(a) $3 \times 231 = 3 \times (200 + 30 + 1)$

$= (3 \times 200) + (3 \times \underline{30}) + (3 \times 1)$

$= \underline{600} + \underline{90} + \underline{3}$ or $\boxed{693}$

(b) $4 \times 122 = 4 \times (100 + 20 + \underline{2})$

$= (4 \times 100) + (4 \times \underline{20}) + (4 \times 2)$

$= 400 + \underline{80} + \underline{8}$ or $\boxed{488}$

(c) $8 \times 322 = 8 \times (\underline{300} + \underline{20} + \underline{2})$

$= (8 \times 300) + (8 \times 20) + (8 \times 2)$

$= \underline{2400} + \underline{160} + \underline{16}$ or $\boxed{2576}$

Pupil Exercises. Provide each pupil with worksheet No. 19.

Limit the explanation of this worksheet with the more able pupils. With the less able group, provide sufficient practice and assistance before independent work is assigned.

Worksheet No. 19

FOR EACH OF THE FOLLOWING EXAMPLES, FILL IN THE MISSING NUMERALS:

(a)

1. 5 × 23 = 5 × (20 + 3)
= (5 × 20) + (5 × 3)
= 100 + 15
= 115

(b)

7 × 34 = 7 × (30 + 4)
= (7 × 30) + (7 × 4)
= 210 + 28
= 238

2. 9 × 47 = 9 × (40 + 7)
= (9 × 40) + (9 × 7)
= 360 + 63, or 423

12 × 63 = 12 × (60 + 3)
= (12 × 60) + (12 × 3)
= 720 + 36, or 756

3. 6 × 98 = 6 × (90 + 8)
= (6 × 90) + (6 × 8)
= 540 + 48, or 588

25 × 26 = 25 × (20 + 6)
= (25 × 20) + (25 × 6)
= 500 + 150, or 650

4. 5 × 327 = 5 × (300 + 20 + 7)
= (5 × 300) + (5 × 20) + (5 × 7)
= 1,500 + 100 + 35, or 1,635

5. 8 × 451 = 8 × (400 + 50 + 1)
= (8 × 400) + (8 × 50) + (8 × 1)
= 3,200 + 400 + 8, or 3,608

6. 15 × 122 = 15 × (100 + 20 + 2)
= (15 × 100) + (15 × 20) + (15 × 2)
= 1,500 + 300 + 30, or 1,830

DRAW A CIRCLE AROUND THE LETTER OF EACH EXAMPLE FOR EACH TRUE SENTENCE.

7. [✓ (a)] 24 + 5 = 5 + 24
(b) (8 × 7) < (11 × 5)
[✓ (c)] (5 × 9) < (12 × 4)
[✓ (d)] 63 + 63 + 63 + 63 = (126 × 2)
[✓ (e)] $14_{\text{five}} = 1_{\text{five}} + 4_{\text{ones}}$
[✓ (f)] 35_{six} = 3 sixes and 5 ones

5. Division and Multiplication Relationship

Objective. To review and extend the meaning of division using sets, arrays, and number sentences.

Background Information for the Teacher. Division of whole numbers is defined as a binary operation in which two given whole numbers are combined to produce a third whole number called the quotient. There are examples, however, when it is not possible to produce a third whole number.

Just as multiplication is thought of as repeated addition, division is sometimes thought of as repeated subtraction. Consider for a moment the example, $12 \div 3 = \boxed{}$. The result of dividing the pair of numbers 12 and 3 begins with 12. Subtracting 3 from 12 leaves 9; subtracting 3 from 9 leaves 6; subtracting 3 from 6 leaves 3; etc. We continue to subtract 3 until we get a difference of 0. The number of times that 3 has been subtracted from 12 is the result of 12 ÷ 3, in this case, 4 (see illustration).

$$\begin{array}{rr} 12 & \\ -\ 3 & 1 \\ \cline{1-1} 9 & \\ -\ 3 & 1 \\ \cline{1-1} 6 & \\ -\ 3 & 1 \\ \cline{1-1} 3 & \\ -\ 3 & 1 \\ \hline 0 & 4 \end{array}$$

This discovery permits us to design an algorithm to speed up the search for the quotient. Thus, we may arrange our numerals as follows:

$$\begin{array}{r|r} 3\overline{)\ 12} & \\ -\ 3 & 1 \\ \cline{1-1} 9 & \\ -\ 3 & 1 \\ \cline{1-1} 6 & \\ -\ 3 & 1 \\ \cline{1-1} 3 & \\ -\ 3 & 1 \\ \hline 0 & 4 \end{array}$$

When the pupil knows his multiplication combinations well and has had an opportunity to analyze many examples, he will recognize at once that 3 will be subtracted repeatedly, 4 times. Thus, he may use this algorithm:

$$\begin{array}{r} 4 \\ 3\overline{)\ 12} \\ -12 \\ \hline 0 \end{array}$$

If we consider the pair of numbers 36 and 5, we will recognize that the quotient is not a whole number but a fractional number, $7\frac{1}{5}$, as shown in the illustrations (a) and (b) below:

(a)

$$36 \div 5 = 7\frac{1}{5}$$

(b)

$$\begin{array}{r} 7 \\ 5\overline{)\ 36} \\ -35 \\ \hline 1 \end{array}$$

Interpreting division as repeated subtraction does not always give us a quotient which is in the set of whole numbers. However, the repeated subtraction concept does promote the understanding of the division algorithm.

To study the operation of division, let's look to multiplication to see if a relationship exists. For example, by examining illustrations (a) and (b) below:

(a)

Factor		Factor		Product
3	×	4	=	12
4	×	3	=	12

(b)

Product		Factor		Factor
12	÷	3	=	4
12	÷	4	=	3

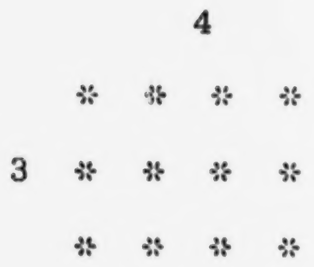
We can see that the "known" factors in multiplication can satisfy the "missing factors" in division. For example:

If $3 \times 4 = 12$, then $12 \div 3 = 4$ because $3 \times 4 = 12$;

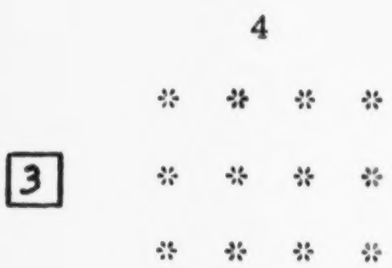
If $3 \times 4 = 12$, then $12 \div 4 = 3$ because $4 \times 3 = 12$.

Using an array, we might represent 12 objects as 3 sets with 4 members in each set, as in illustration (a):

(a)



(b)



In illustration (b) we know that there are 12 objects, one factor is "known," that is, 4; we are looking for the "missing factor" which is 3.

TEACHING PROCEDURE

Exploration. On the chalkboard write two related multiplication sentences, such as 5 × 6 = 30 and 6 × 5 = 30. Now ask, "If five 6's are equal to 30, how many 6's are in 30? 5. If six 5's are equal to 30, how many 5's are in 30?" 6.

Next, illustrate a 3 × 6 array and ask, "How many sets of 3 are

✼ ✼ ✼ ✼ ✼ ✼

✼ ✼ ✼ ✼ ✼ ✼

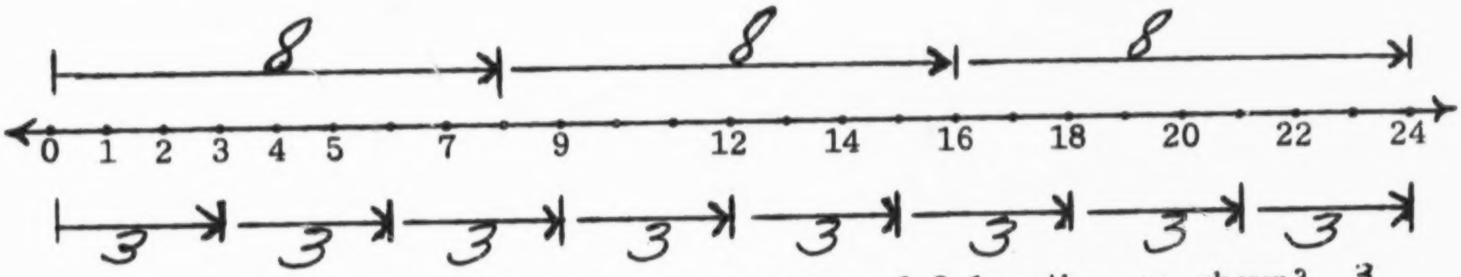
✼ ✼ ✼ ✼ ✼ ✼

there in 18? 6. How many sets of 6 are there in 18? 3. What is the product of 3 × 6 = $\boxed{18}$? Name the missing factor in 18 ÷ 3 = $\boxed{6}$

Next, write the two multiplication sentences and the two division sentences for the 3 × 6 array shown above.

Multiplication	Division
3 × 6 = 18	18 ÷ 3 = 6
6 × 3 = 18	18 ÷ 6 = 3

Use the number line to increase understanding between multiplication and division. Illustrate the following on the chalkboard and have pupils identify the measure of each segment, then ask the following questions:



(1) "On the number line, how many moves of 8 lengths are shown? 3
Write a multiplication sentence to show three 8's." 3 X 8 = 24

(2) "If we begin at 24 on the number line and move to the left, how many moves of 8 lengths can we make to get back to 0 ? 3 Write a division sentence to show this." 24 ÷ 8 = 3

(3) "How many moves of 3 lengths are shown on the number line?" 8
Write a multiplication sentence to show eight 3's." 8 X 3 = 24
Write a division sentence to show this." 24 ÷ 3 = 8

On the chalkboard write the following examples for individual practice. Inform the pupils that the four sentences in each row are related; knowing the solution can be helpful in solving the others.

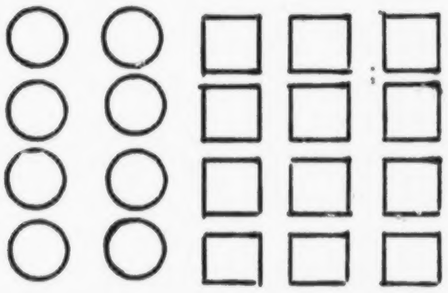
(1) 6 × 3 = 18   18 ÷ 3 = 6   3 × 6 = 18   18 ÷ 6 = 3

(2) 2 × 8 = 16   16 ÷ 8 = 2   8 × 2 = 16   16 ÷ 2 = 8

(3) 14 ÷ 7 = 2   2 × 7 = 14   14 ÷ 2 = 7   7 × 2 = 14

Before introducing division and addition to discover some new facts, review the distributive principle of multiplication over addition. Since multiplication and division are related, separating a factor into two addends will be employed in division.

On the chalkboard, illustrate a 4 × 5 array such as the one below.*



Now ask the pupils, "How many objects are represented in each row? 5 How many objects are represented in each column? 4 Write a multiplication sentence to show how many objects are represented in this complete array. 4 × 5 = 20 Is it possible to show a division sentence using the numbers 4, 5, and 20? Yes, 20 ÷ 5 = 4; 20 ÷ 4 = 5. Does the sentence 20 ÷ 4 = ☐ tell us something about the illustration? Yes, we are looking for the missing factor which is 5.

Next, guide the pupils to find the missing factor in the following by asking:

(1) "How many circles are represented in our illustration? 8 How many rows of circles are there? 4 To find the number of members in each row of circles (the missing factor), what must we do?" Divide; 8 ÷ 4 = 2; there are 2 members in each set of circles.

(2) "How many squares are represented in our illustration? 12 How many rows of squares are there? 4 To find the number of members in each row of squares (the missing factor), what must we do?" Divide; 12 ÷ 4 = 3; there are three members in each set of squares.

---

* The use of the flannel board and cutouts may be helpful for this concept. For example, the teacher may wish to use 4 rows of geometric shapes; 2 columns of circles and 2 columns of squares.

(3) "Does our illustration show that we have 8 circles and 12 squares? Yes Is $(8 + 12)$ another name for 20? Yes Then, what can we say about this sentence?"

$$(8 + 12) \div 4 = (8 \div 4) + (12 \div 4)$$

This sentence is true; 20 is renamed so that we can divide twice which is easier than dividing once, especially if you do not know the combinations.

It is unlikely that the pupils will see that the divisor is distributed to 8 and 12 with respect to addition. However, the similarity between the distributive principle of multiplication over addition and that of division should be pointed out. This principle has advantages; if the dividend is too large and the division-multiplication combinations are not completely mastered, the dividend may be renamed and the divisor distributed.

Since this is an advanced concept, many illustrations and examples will have to be presented, proceeding from the concrete (objects) to the semi-abstract (symbols).

Next, point out to the pupils that in order to find the missing factor in this example, $25 \div 5 = \square$, it is possible to choose another name for 25 which is convenient for simplifying division. For example, 25 may be renamed as: $(15 + 10)$, $(20 \div 5)$, $(10 + 15)$, etc. On the chalkboard, illustrate the following:

$$25 \div 5 = \square$$

$$(15 + 10) \div 5 = (15 \div 5) + (10 \div 5)$$

$$= \underline{\ 3\ } + \underline{\ 2\ }, \text{ or } \boxed{5}$$

To analyze what has actually taken place, ask these and similar questions: *

(1) "How have we renamed 25? $(15+10)$ How many times have we divided? Twice: $(15 \div 5)$ and $(10 \div 5)$ What is the missing factor for $15 \div 5 = \square$? 3 What is the missing factor for $10 \div 5 = \square$? 2 What is the missing factor for $25 \div 5 = \square$?" 5

(2) "How does the missing factor for $25 \div 5 = \square$ compare with the sum of the missing factors for $(15 \div 5) + (10 \div 5)$? They are equal. Does $25 \div 5 = (15 \div 5) + (10 \div 5)$?" Yes.

---

*Record the pupils' responses in each blank space in the illustration.

(3) "Would renaming 25 as (13 + 12) be convenient to simplify division?"

No, because $(13 \div 5) + (12 \div 5)$ would not name whole numbers.

For pupil practice, write the following examples on the chalkboard. Have some pupils work at the chalkboard while others work at their desks. Lead them to increase their understanding of the distributive idea by having them fill in the missing numerals.

(a) $30 \div 5 = \boxed{6}$; $30 \div 6 = \boxed{5}$; $5 \times 6 = \boxed{30}$; $6 \times 5 = \boxed{30}$

(b) $\boxed{6} \times 5 = 30$; $6 \times \boxed{5} = 30$; $30 \div \boxed{6} = 5$; $30 \div \boxed{5} = 6$

(c) $28 \div 4 = (20 \div 4) + (8 \div 4)$
$= \underline{5} + \underline{2}$, or $\boxed{7}$

(d) $24 \div 3 = (18 \div 3) + (6 \div 3)$
$= \underline{6} + \underline{2}$, or $\boxed{8}$

(e) $30 \div 5 = (15 \div 5) + (15 \div 5)$
$= \underline{3} + \underline{3}$, or $\boxed{6}$

Pupil Exercises. Provide each pupil with worksheet No. 20.

The pupils will need a ruler or straightedge to do examples 5 through 8. Encourage neat and accurate work. With less able pupils, some of these examples may be worked cooperatively at the chalkboard.

Worksheet No. 20

WRITE THE MISSING NUMERALS IN THE BLANK SPACES FOR EACH OF THE FOLLOWING:

(a)

1. $96 \div 8 = (80 + 16) \div 8$
$= (80 \div 8) + (16 \div 8)$
$= 10 + 2$, or 12

2. $645 \div 15 = (600 + 45) \div 15$
$= (600 \div 15) + (45 \div 15)$
$= 40 + 3$, or 43

3. $625 \div 25 = (500 + 125) \div 25$
$= (500 \div 25) + (125 \div 25)$
$= 20 + 5$, or 25

(b)

$225 \div 5 = (200 + 25) \div 5$
$= (200 \div 5) + (25 \div 5)$
$= 40 + 5$, or 45

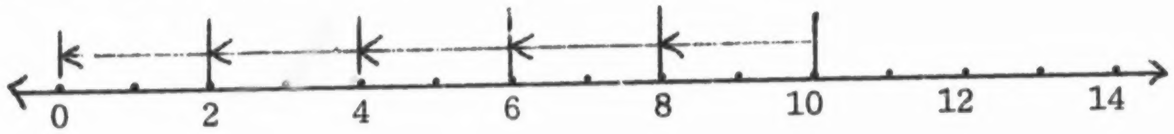
$46 \times 8 = (40 + 6) \times 8$
$= (40 \times 8) + (6 \times 8)$
$= 320 + 48$, or 368

$13 \times 66 = 13 \times (60 + 6)$
$= (13 \times 60) + (13 \times 6)$
$= 780 + 78$, or 858

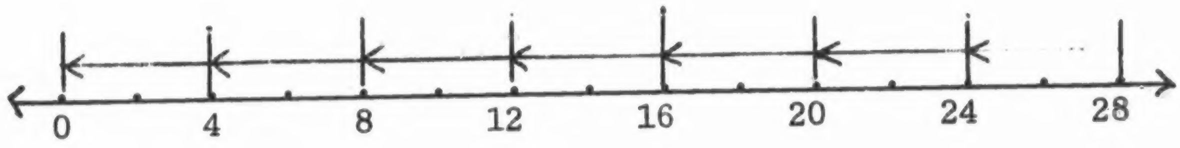
4. $850 \div 50 = (400 + 400 + 50) \div 50$
$= (400 \div 50) + (400 \div 50) + (50 \div 50)$
$= 8 + 8 + 1$, or 17

ILLUSTRATE THE FOLLOWING NUMBER SENTENCES ON THE NUMBER LINE:*

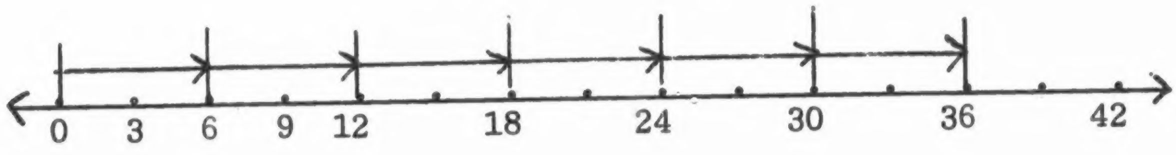
5. $10 \div 2 = 5$



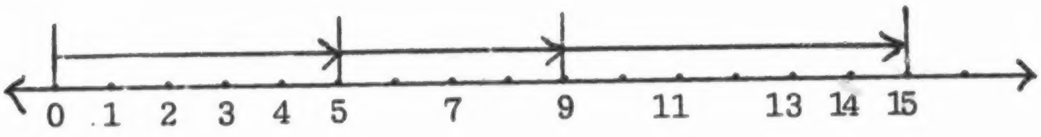
6. $28 \div 4 = 7$



7. $6 \times 6 = 36$



8. $(5 + 4) + 6 = 15$



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*Illustrations on the number line will vary.

6. Dividing Tens and Ones Separately

Objective. To develop skill in renaming a dividend into two parts so that it is convenient to divide the parts separately, using the distributive idea.

Background Information for the Teacher. It is suggested that the teacher present many examples to the class, making certain that pupils understand they are to add the two partial quotients after dividing the parts separately. For example, the conventional algorithm, illustration (a), can be somewhat mechanical; illustration (b) renames 96 as 9 tens and 6 ones or 90 + 6; illustration (c) uses the horizontal notation, incorporating the distributive principle and adding the partial quotients.

(a)

$$\begin{array}{r} 32 \\ 3\overline{)\,96} \\ \underline{9} \\ 6 \\ \underline{6} \end{array}$$

(b)

$$\begin{array}{l} 30 + 2 = 32 \\ 3\overline{)\,90 + 6} \end{array}$$

(c)

$$\begin{aligned} 96 \div 3 &= \square \\ (90 + 6) \div 3 &= (90 \div 3) + (6 \div 3) \\ &= 30 + 2 \\ &= 32 \end{aligned}$$

It may be necessary, especially with some pupils, to use the place-value chart or bundles of sticks to represent tens and ones to illustrate the renaming of the dividend.

TEACHING PROCEDURE

Exploration. On the chalkboard, provide pupils with examples such as the following which will give practice for renaming the dividend as tens and ones. (Later it will be seen that the specific parts selected will depend on the divisor.)

96 = 90 + 6

86 = 80 + 6

48 = 40 + 8

66 = 60 + 6

Next, write the following problem on the chalkboard:

> "Bob has 96 soldiers in his Battleground game. He wants to put them into 3 boxes, with the same number in each box. How many should he put in each box?"

Ask, "What division sentence should we write for this problem? $96 \div 3 = \square$ To find the missing numeral, can we rename 96 as (90 + 6)? Yes Would it be convenient to rename 96 as (92 + 4)?" No, because we are trying to show tens and ones in preparation for division.

Write the following example on the chalkboard and ask the pupils to name the missing numerals:

$$96 \div 3 = \square$$

$$(90 + 6) \div 3 = (90 \div 3) + (6 \div 3)$$

$$= \underline{30} + \underline{2}, \text{ or } \boxed{32}$$

As the pupils respond to the following questions, write in the correct responses: Ask, "What numerals do we write in the blank spaces? 30 and 2 What is 96 ÷ 3? 32 How many soldiers should Bob put in each box?" 32

Use another division sentence such as $84 \div 2 = \square$. Now ask the pupils to tell a story which fits the example. A possible story might be:

> "Four third-grade classrooms from Hawthorne Elementary School went to Coronado. There were 84 children in all. Two buses were provided for the trip. How many children rode in each bus if each bus had the same number of children?"

Have the pupils respond to questions so that the problem may be stated as:

$$84 \div 2 = \square$$

$$(80 + 4) \div 2 = (80 \div 2) + (4 \div 2)$$

$$= \underline{40} + \underline{2}, \text{ or } \boxed{42}$$

Ask, "What numerals do we write in the blank spaces? 40 and 2 What is the sum of $(40 + 2) = \square$? 42 What is 84 ÷ 2 ? 42 .

Provide the pupils with practice so that they may develop skill in thinking of the dividend as tens and ones, adding the partial quotients, and getting the solution. Write these and similar examples on the board:

(a) $44 \div 2 = (40 \div 2) + (4 \div 2)$

$$= \underline{20} + \underline{2}, \text{ or } \underline{22}$$

(b) $48 \div 4 = (40 \div 4) + (8 \div 4)$

$= \underline{10} + \underline{2}$, or $\underline{12}$

(c) $95 \div 5 = (90 \div 5) + (5 \div 5)$

$= \underline{18} + \underline{1}$, or $\underline{19}$

(d) $77 \div 7 = (70 \div 7) + (7 \div 7)$

$= \underline{10} + \underline{1}$, or $\underline{11}$

In discussing each example, write in the correct responses as the pupils respond to questions regarding the missing numerals.

Whenever the pupils begin to understand division in terms of thinking about the dividend as tens and ones, extend the concept to include higher decade division. For example, write the following problem on the chalkboard:

> "Sarah has 128 pencils to share with 4 friends. She plans to give each friend the same number of pencils. How many pencils will each friend get?"

Next, ask the pupils if the following sentence tells the story about the problem:

$$128 \div 4 = \square$$

$$(120 + 8) \div 4 = (120 \div 4) + (8 \div 4)$$

$$= \underline{30} + \underline{2}, \text{ or } \boxed{32}$$

Ask, "How did we rename 128? $\underline{(120 + 8)}$ How many tens and ones are represented by (120 + 8)? $\underline{\text{12 tens and 8 ones}}$ What is (120 ÷ 4)? $\underline{30}$ What is (8 ÷ 4)? $\underline{2}$ What is (128 ÷ 4)? $\underline{32}$ Which of the following are true?"

(a) $128 \div 4$ is equal to $(120 \div 4) + (8 \div 4)$ True

(b) $128 = 120 + 8$ True

(c) $32 \times 4 = (30 \times 4) + (2 \times 4)$ True

(d) $32 \times 4 = 128$ True

(e) $128 \div 4 = 32$ True

Write the following examples on the chalkboard for independent practice at desks:

(a) $255 \div 5 = (250 \div 5) + (5 \div 5)$

$= \underline{25} + \underline{1}$, or $\underline{26}$

(b) $208 \div 4 = (200 \div 4) + (8 \div 4)$

$= \underline{50} + \underline{2}$, or $\underline{52}$

(c) $126 \div 6 = (120 \div 6) + (6 \div 6)$

$= \underline{20} + \underline{1}$, or $\underline{21}$

Encourage the pupils to look for a pattern in determining tens and ones and to work independently. Do not hesitate to give help. Pupils sometimes seem to lose the idea for a moment, then a word at the right time will put them on the track.

Pupil Exercises. Provide each pupil with worksheet No. 21.

Limit the explanation of this worksheet with the more able group. With the less able group, offer as much explanation as you feel is necessary. Problems 1 through 4 may be worked cooperatively with some pupils. Encourage variety in establishing number sentences to solve the problems.

Interim Mathematics Unit - Grade 6
San Diego City Schools (1-65)

Worksheet No. 21

WRITE A COMPLETE NUMBER SENTENCE FOR EACH PROBLEM. DECIDE WHETHER TO ADD, SUBTRACT, MULTIPLY, OR DIVIDE.

On March 27, 1964, the state of Alaska had a severe earthquake. Many buildings were destroyed, hundreds of people were injured, and some lives were lost.

1. To clear the damage in one city, 35 cranes, 15 bulldozers, and 49 trucks were used. How many items of equipment were used to clear the damage?

$(35 + 15) + 49 = 99$

2. To transport the injured to temporary hospitals, an ambulance made 18 trips in one day. On each trip, 6 people occupied the ambulance. How many people were transported by this ambulance in one day?

$6 \times 18 = 6 \times (10 + 8)$

$= (6 \times 10) + (6 \times 8)$

$= 60 + 48$, or 108

3. Another ambulance transported a total of 48 people in one day. Six people were transported on each trip. How many trips did the ambulance make?

$48 \div 6 = (30 + 18) \div 6$

$= (30 \div 6) + (18 \div 6)$

$= 5 + 3$, or 8

4. One truck driver hauling debris traveled 855 miles during the first week and 676 miles the second week. What was the difference in the miles driven during the two weeks?

$855 - 676 = 179$

WRITE THE MISSING NUMERAL IN THE BLANK SPACES.

5. 459 ÷ 9 = (450 ÷ 9) + (9 ÷ 9)

= 50 + 1, or 51

6. 225 ÷ 25 = (200 ÷ 25) + (25 ÷ 25)

= 8 + 1, or 9

7. 856 ÷ 8 = (800 ÷ 8) + (56 ÷ 8)

= 100 + 7, or 107

8. 465 ÷ 5 = (400 ÷ 5) + (50 ÷ 5) + (15 ÷ 5)

= 80 + 10 + 3, or 93

7. Introducing the Long-division Form

Objective. To develop a division algorithm which is fundamental to future work and leads to a better understanding of the conventional algorithm.

Background Information for the Teacher. The approach to division in this section will be used consistently throughout this unit until the conventional algorithm is employed. However, as the pupils make further progress in dividing tens and ones separately in this grade and later grades, they will need to learn the familiar algorithm which will eventually be used in higher-decade division.

Interpreting division as repeated subtraction as indicated previously (see page 109), does not always give us a quotient which is in the set of whole numbers. However, the repeated subtraction concept does promote the understanding of the division algorithm as illustrated by the story which follows.

Suppose for a moment that three different pupils worked on the example $136 \div 8 = \square$ using the repeated subtraction concept. In analyzing examples (a), (b), and (c) representing the work of each pupil, we would probably state that:

(a)

$$\begin{array}{r|l} 8\,\overline{)\,136} & \\ -\underline{40} & 5 \\ 96 & \\ -\underline{40} & 5 \\ 56 & \\ -\underline{40} & 5 \\ 16 & \\ -\underline{8} & 1 \\ 8 & \\ -\underline{8} & \underline{1} \\ 0 & 17 \end{array}$$

(b)

$$\begin{array}{r|l} 8\,\overline{)\,136} & \\ -\underline{80} & 10 \\ 56 & \\ -\underline{40} & 5 \\ 16 & \\ -\underline{16} & \underline{2} \\ 0 & 17 \end{array}$$

(c)

$$\begin{array}{r|l} 8\,\overline{)\,136} & \\ -\underline{80} & 10 \\ 56 & \\ -\underline{56} & \underline{7} \\ 0 & 17 \end{array}$$

Pupil (a) is just beginning his study of the division algorithm; pupil (b) is beginning to develop a greater insight into the division algorithm; and pupil (c) is most familiar with the division algorithm. In all three examples, the pupils understand division. After a good development of the division algorithm, all three pupils should write the familiar algorithm, that is:

$$\begin{array}{r} 17 \\ 8\,\overline{)\,136} \\ -\underline{80} \\ 56 \\ -\underline{56} \\ 0 \end{array}$$

In the new state textbooks, the long form of division is developed and retaught through understanding rather than by requiring pupils to memorize a series of steps. This division form requires the use of place-value and the ability to make estimates of the trial quotient. However, the more accurately a pupil estimates the trial quotient, the shorter his computation will be.

It is likely that pupils in the class have been introduced to the division process in the conventional manner. As a consequence, some pupils may not be quick to gain an understanding of the new method. Some "unlearning" may be necessary.

TEACHING PROCEDURE

Exploration. To introduce the "new" division algorithm, illustrate the following example (Figure 1) on the chalkboard. Ask the pupils to analyze the division example and be prepared to discuss the process involved. In the event pupils are reluctant to discuss the process or perhaps are not "sure," ask the following questions:

Figure 1

```
      114
      ???
 5 ) 570 |
     500 |    100    (a)
      70 |
      50 |     10    (b)
      20 |
      20 |      4    (c)
         |
         |    114
```

(1) "Are there at least 100 sets of 5 in 570? Yes Are there as many as 200 sets of 5 in 570? No Since there are at least 100 sets of 5 in 570, let's write 100 at the side [see (a)]."

(2) "What is the product of 100 × 5? 500 If we subtract the 500 from 570, what is the difference? 70 How many sets of 5 are contained in 70--are there at least 5? yes At least 10? yes As many as 15? No What is the product of 10 × 5? 50 Since there at at least 10 more sets of 5 in 70, let's write 10 at the side [see (b)]."

(3) "If we subtract 50 from 70, what is the difference? 20 How many sets of 5 are there in 20? 4 Let's write the 4 at the side [see (c)]. What is the product of 4 × 5? 20 If we subtract 20 from 20, is there anything left to divide?" No

(4) "Now let's add the partial quotients: 100 + 10 + 4 = 114 Is 114 the quotient for the example (Figure 1)? yes We write the quotient above the dividend, where the question marks are.* Check the answer by multiplying: 114 × 5 = 570."

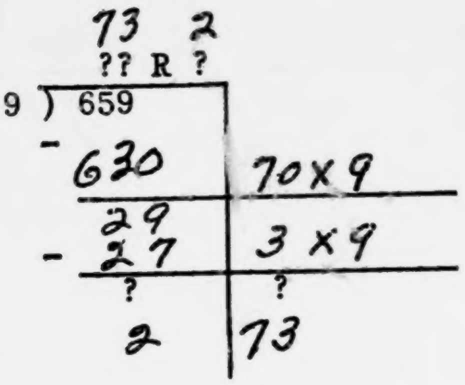
(5) "Examine the work we have done. We have found that (100 × 5) + (10 × 5) + (4 × 5) = 570. In other words, 570 has been regrouped as 500 + 50 + 20."

---

*Record the quotient, replacing the question marks.

Next, use the division algorithm to analyze an example which involves a quotient having a remainder (Figure 2).

Figure 2



Ask the following questions:

(1) "First, let's try to find the largest number of nines in 659. Are there as many as 100 sets of nine in 659? No Then let's begin by thinking about tens. Are there as many as 50 nines in 659? yes As many as 60? yes As many as 70? yes As many as 80? No In which of these is there the largest number of nines in 659 that we have found so far? What is the product of 70 × 9?" $\boxed{630}$*

(2) "What is the difference when we subtract 630 from 659? 29* How many sets of nine are contained in 29? 3* What is the product of 3 × 9? 27* What is 29 - 27? 2* What do we call the 2 we have remaining? remainder What is the sum of 70 + 3? 73* In an example such as this, we say we have a quotient of 73 with a remainder of ___ The complete quotient is written 73 R 2."

At this point, emphasize that division is "checked" by multiplying the divisor by the quotient and adding the remainder. By using the number sentence form (Figure 3), pupils will see the relationship of the quotient and divisor to the dividend.

Figure 3

$$(\boxed{73} \times 9) + \triangle\, 2 = 659$$

Ask the following questions:

(1) "In the division example (Figure 2), can the results be checked just by multiplying 73 × 9 = ☐? No What step is missing? The remainder 2 must be added.

(2) "Suppose we use a ☐ to represent the quotient and a △ to represent the remainder (Figure 3). Can our result be checked, using this sentence? yes What is the product of 73 × 9? 657 What is the sum of 657 + 2? 659 Does (73 × 9) + 2 = 659?" yes

For individual or group practice, write the following examples on the chalkboard. The pupils' completed work should appear similar to that which is shown, except for example (f):

---

*Record the responses to complete the algorithm.

(a)

```
     178
5 ) 890 |
    500 | 100
    390 |
    350 |  70
     40 |
     40 |   8
        | 178
```

(b)

```
     131
7 ) 917 |
    700 | 100
    217 |
    210 |  30
      7 |
      7 |   1
        | 131
```

(c)

```
     112
8 ) 896 |
    800 | 100
     96 |
     80 |  10
     16 |
     16 |   2
        | 112
```

Divide. Check all answers.

(d)

```
    43 R1
9 ) 388 |
    360 | 40
     28 |
     27 |  3
      1 | 43
```

$(43 \times 9) + 1 = 387 + 1$
$= 388$

(e)

```
    38 R5
8 ) 309 |
    240 | 30
     69 |
     64 |  8
      5 | 38
```

$(38 \times 8) + 5 = 304 + 5$
$= 309$

(f)

```
    54 R5
6 ) 329 |
    240 | 40
     89 |
     60 | 10
     29 |
     24 |  4
      5 | 54
```

$(54 \times 6) + 5 = 324 + 5$
$= 329$

The pupils should be encouraged to strive for efficiency in all of their work. For example, in (f) above, a pupil may say there are at least 40 sets of six in 329 and proceed as shown. However, though there is nothing wrong with this procedure, he should strive for the largest number of sets of six (tens of sixes [50]) in the first step in order to eliminate unnecessary steps in finding the quotient.

Next, point out to the pupils that they have been investigating some ideas that are used in division. Since division is not a new topic for them, discuss also the relationship between the "new" division algorithm and the conventional or "shorter form" with which they are familiar. As an illustration, write the following examples (Figures 4 and 5) on the chalkboard.

Figure 4

```
      23 R 9
41 ) 952 |
     820 |  20
     132 |
     123 |   3
       9 |  23
```

Figure 5

```
      23 R 9
41 ) 952
     82
     132
     123
       9
```

Ask the following questions:

(1) "In Figure 4, are there as many as 100 sets of forty-one contained in 952? No Then we must think of tens instead of hundreds. Are there as many as 10 sets of forty-one in 952? Yes As many as 20 sets of forty-one? Yes As many as 30 sets of forty-one? No Then, what is the largest number of 41's in 952? 20 Why do we write the 20 at the side?

To keep a record of the number of sets of forty-one contained in 952. What is the product of 41 × 20? 820 Why do we subtract 820 from 952? To see if there are additional sets of forty-one in the difference. How many sets of forty-one are there in 132? 3 What is the product of 41 × 3? 123 Why do we subtract 123 from 132? To see if there are additional sets of forty-one in the difference. What is the quotient? 23 The remainder?" 9

(2) "In the shorter form (Figure 5), when we find there are 20 sets of forty-one--but not 30 sets of forty-one-- in 952, do we know that the quotient is some number in the 20's? Yes Where do we write the 2 (which represents 2 tens)? above the tens' place When we multiply 2 × 41 = 82, what do we actually mean? The 2 really means 20; thus, the product 82 means 820. When we subtract 82 and bring down the 2 from the dividend, what are we actually doing? We are subtracting 820 from 952. Next, we find that there are 3 more sets of forty-one in 132. Where do we write the 3? above the ones' place in the dividend. How do we get the number 123? 41 × 3 The remainder?" 132-123.

Use other examples to compare the "new" algorithm and the conventional or shorter form. Discuss their relationship. To insure greater understanding, it may be necessary to work a few of the examples cooperatively, then assign these or similar examples for practice. Have the pupils show their work using both forms, the "new" algorithm and the shorter form.

(a)	(b)	(c)
75 R 10	9 R 32	50 R 12
50) 3760	60) 572	23) 1162

Pupil Exercises. Provide each pupil with worksheet No. 22.

In working the division examples in No. 4, permit the pupils to use the method they feel is easiest for them. Undoubtedly, many will rely on the conventional or shorter form as opposed to that of the "new" division algorithm. However, encourage frequent use of the "new" algorithm since it is the basis for the ideas contained in the shorter form.

DRAW A CIRCLE AROUND THE LETTER FOR EACH EXAMPLE WHICH IS TRUE.

1. [✓ (a)] $75 \times 5 < 400$ (d) $8 \times 60 > 500$ [✓ (g)] $(40 \times 2) = (2 \times 40)$
[✓ (b)] $7 \times 70 < 510$ [✓ (e)] $9 \times 80 > 675$ [✓ (h)] $(85 + 15) > (15 + 84)$
[✓ (c)] $626 > (25 \times 25)$ [✓ (f)] $6 \times 90 < 552$ [✓ (i)] $(8 \times 46) < 372$

WORK EACH EXAMPLE USING TWO DIFFERENT FORMS. IN EXAMPLE 2 (a), (b), and (c) WRITE THE QUOTIENT ABOVE THE DIVIDEND.

2. (a) 7) 637, quotient 91
630 = $\boxed{90}$ × 7
7
7 = $\boxed{1}$ × 7
0

(b) 25) 625, quotient 25
500 = $\boxed{20}$ × 25
125
125 = $\boxed{5}$ × 25
0

(c) 29) 246, quotient 8 R14
232 = $\boxed{8}$ × 29
14

3. (a) 7) 637, quotient 91
63
7
7
0

(b) 25) 625, quotient 25
50
125
125
0

(c) 29) 246, quotient 8 R 14
232
14

USE EITHER FORM TO SOLVE THE FOLLOWING:

4. (a) 28) 1722, quotient 61 R 14
(b) 37) 2109, quotient 57
(c) 27) 8115, quotient 300 R15

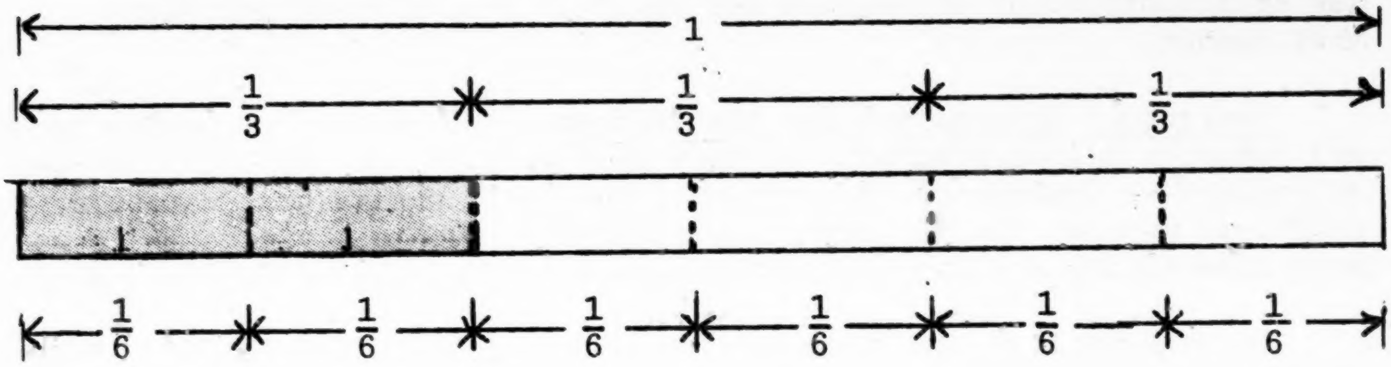
LET'S PRACTICE. (Use the largest number possible to replace each frame.)

5. (a) $\boxed{6} \times 57 < 376$ (c) $45 \times \boxed{8} < 405$ (e) $\boxed{9} \times 28 = 260$
(b) $\boxed{6} \times 77 < 470$ (d) $\boxed{40} \times 38 = 1,535$ (f) $\triangle\,10 \times 15 < 153$

WRITE THE SYMBOL =, >, or < IN EACH BLANK SPACE TO MAKE THE SENTENCE TRUE.

6. (a) $89 + 47 > 78 + 53$ (d) $129 \div 3 > 164 \div 4$
(b) $127 + 354 < 427 + 64$ (e) $568 \div 4 > 272 \div 2$
(c) $6 \times 9 < 8 \times 7$ (f) $6 \times 34 > 5 \times 38$

Figure 2



When $\frac{2}{6}$ is renamed as $\frac{1}{3}$, we say that it is in lowest terms. A fraction is said to be in lowest terms if the numerator and the denominator name no common factor other than 1. In showing that $\frac{2}{6}$ may be reduced to $\frac{1}{3}$, we see that

$$\begin{aligned} \frac{2}{6} &= \frac{2}{6} \div \frac{2}{2} \\ &= \frac{2 \div 2}{6 \div 2} \\ &= \frac{1}{3} \end{aligned}$$

Ask the following questions:

(1) "In the illustration (Figure 2), what part of the basic region is not shaded? $\underline{\frac{4}{6}}$ What is another name for $\frac{4}{6}$?" $\underline{\frac{2}{3}}$

(2) "If the entire basic region were shaded, how many sixths would be represented? $\underline{\frac{6}{6}}$ What is another name for $\frac{6}{6}$?" $\underline{1}$

2. Extending Equivalent Fractions

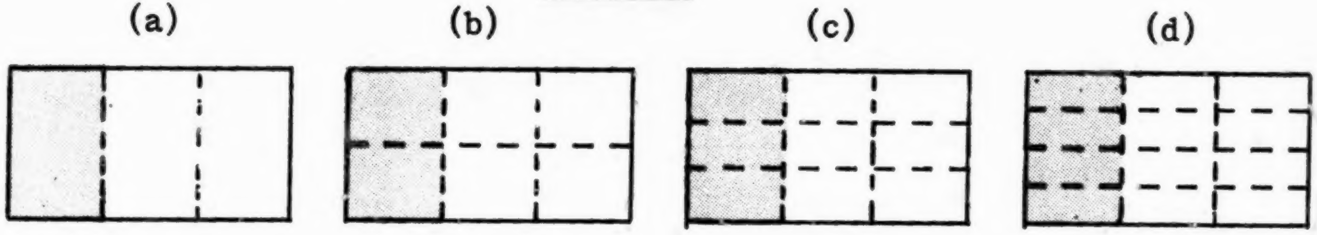
Objective. To extend the concept of equivalent fractions through the use of illustrations of regions and the fact that 1 times any number is that number.

Background Information for the Teacher. The ability of pupils to add and subtract unlike fractions depends upon a clear and thorough understanding of equivalent fractions. Too often in the past, pupils have manipulated fractions without the understanding which will eventually make their work more interesting and computation easier.

In this section, two methods of finding equivalent fractions are presented. First, by way of a region diagram, pupils will see that a fractional part of the diagram (Figure 1) can have different names such as

$\frac{1}{3}$ (a), $\frac{2}{6}$ (b), $\frac{3}{9}$ (c), $\frac{4}{12}$ (d), etc.

Figure 1



Second, pupils will derive equivalent fractions by making use of the fact that 1 times any number is that number. Also, they will learn to rename 1 in many ways. For example, $1 \times \frac{1}{3} = \frac{1}{3}$; that is, 1 times $\frac{1}{3}$ is still $\frac{1}{3}$. However, new names for $\frac{1}{3}$ can be obtained by renaming 1 as $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$, etc. Thus, $\frac{2}{2} \times \frac{1}{3}$ gives the equivalent fraction $\frac{2}{6}$; $\frac{3}{3} \times \frac{1}{3}$ gives the equivalent fraction $\frac{3}{9}$; etc. Continuing this process will result in the set $\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots\}$

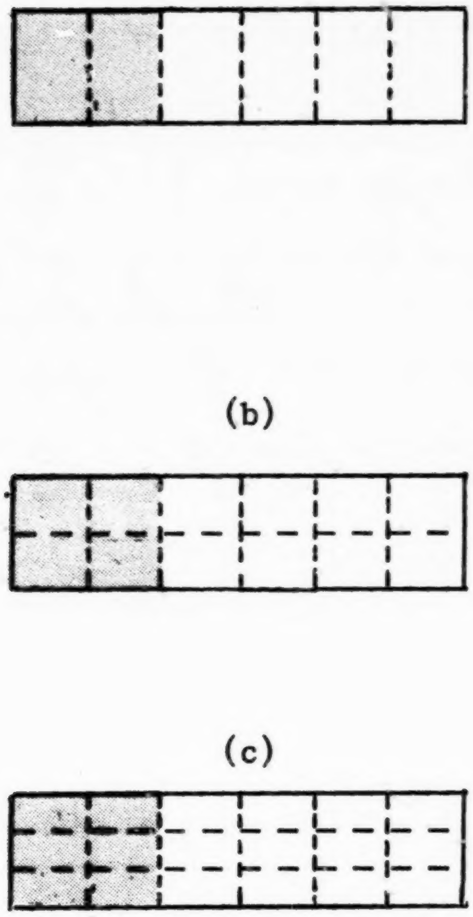
Some pupils may need more experience with series of equivalent fractions. To understand the term "series," you may wish to represent patterns which may provide an opportunity to discover that if the numerator and the denominator of a fraction are multiplied by 2, then by 3, by 4, etc., the result is the series of fractions. As an illustration, study the examples (a) through (d) below:

(a)	(b)	(c)	(d)
$\frac{1}{2} \times \frac{2}{2} = \frac{2}{4}$	$\frac{1}{2} \times \frac{3}{3} = \frac{3}{6}$	$\frac{1}{2} \times \frac{4}{4} = \frac{4}{8}$	$\frac{1}{2} \times \frac{5}{5} = \frac{5}{10}$

TEACHING PROCEDURE

Exploration. On the chalkboard illustrate the rectangular regions as shown in Figure 2, (a) through (c).

Figure 2


(a)

(b)

(c)

Ask the following questions:

(1) "Into how many equal parts is the region (a) divided? 6 What fraction will name one part? $\frac{1}{6}$ How many sixths are shaded? $\frac{2}{6}$ Which term in $\frac{2}{6}$ tells us how many sixths are shaded? the numerator What term names the division of any region?" the denominator

(2) "Into how many equal parts is the region (b) divided? 12 What fraction will name one of the twelve parts? $\frac{1}{12}$ How many twelfths are shaded? $\frac{4}{12}$ In comparing (a) and (b), what true statement can we make?" $\frac{2}{6} = \frac{4}{12}$

(3) "The region (c) has been divided into how many equal parts? 18 What fraction tells how many eighteenths are shaded? $\frac{6}{18}$ Is it true that $\frac{2}{6} = \frac{6}{18}$? Yes That $\frac{4}{12} = \frac{6}{18}$? Yes What true statements can we make about the regions (a), (b), and (c)?"* $\frac{2}{6}$ is equivalent to $\frac{4}{12}$; $\frac{4}{12}$ is equivalent to $\frac{6}{18}$; $\frac{2}{6}$ is equivalent to $\frac{6}{18}$.

After the development of the equivalent fraction concept, using regions, indicate to the pupils that they now have a series of three equivalent fractions: $\frac{2}{6}$, $\frac{4}{12}$, and $\frac{6}{18}$. Have them study this series carefully to find a pattern. Ask, "Name three more equivalent fractions that continue the series." $\frac{5}{15}$, $\frac{7}{21}$, and $\frac{8}{24}$

Next, illustrate Figure 3 on the chalkboard, then ask, "Are $\frac{3}{4}$ and $\frac{18}{24}$ equivalent?" Without any comparison to make, some pupils will answer "No." However, those who have seen the pattern from the illustrations in Figure 2 will undoubtedly answer "Yes." In either case, let the pupils state their reason for their answer. If clarity and restatements are necessary, have them study Figure 3.

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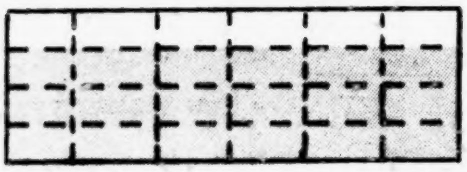
*Answers will vary.

Figure 3



Ask the following questions:

(1) "Is $\frac{3}{4}$ of the illustration shaded? Yes

Is $\frac{18}{24}$ of the illustration shaded? Yes

Then, are $\frac{3}{4}$ and $\frac{18}{24}$ equivalent?" Yes

(2) "In the following series, $\frac{3}{4}$, $\frac{6}{8}$, $\frac{9}{12}$, $\frac{12}{16}$, are all of the fractions equivalent? yes.

Continue the series. $\frac{15}{20}$, $\frac{18}{24}$, $\frac{21}{28}$, $\frac{24}{32}$, etc.

Does the series include $\frac{18}{24}$? Yes

Then is $\frac{18}{24}$ another name for $\frac{3}{4}$?" Yes

Now, have the pupils discover a way of getting directly from $\frac{3}{4}$ to the equivalent fraction $\frac{18}{24}$ without an illustration or a series of equivalent fractions.

Ask the following questions:

(1) "Look at the numerators of the fractions $\frac{3}{4}$ and $\frac{18}{24}$. By what number do we multiply the 3 in $\frac{3}{4}$ to get 18 in $\frac{18}{24}$? 6 Look at the demoninators of the fractions $\frac{3}{4}$ and $\frac{18}{24}$. By what number do we multiply the 4 in $\frac{3}{4}$ to get the 24 in $\frac{18}{24}$? 6 Did we multiply the numerator and the demoninator of $\frac{3}{4}$ by the same number to get $\frac{18}{24}$? Yes Is it true that: $\frac{6}{6} \times \frac{3}{4} = \frac{6 \times 3}{6 \times 4} = \frac{18}{24}$?" Yes

(2) "Express $\frac{6}{6}$ in lowest terms. $\frac{1}{1}$ or 1 What is $1 \times \frac{3}{4}$? $\frac{3}{4}$ What is $\frac{6}{6} \times \frac{3}{4}$? $\frac{18}{24}$ Then is it true that $\frac{1}{1} \times \frac{3}{4} = \frac{3}{4}$ and that $\frac{6}{6} \times \frac{3}{4} = \frac{18}{24}$? Yes Is it true that $\frac{3}{4}$, $\frac{6}{6} \times \frac{3}{4}$, and $\frac{18}{24}$ are different names for the same number?" Yes

Next, use the following illustration to help pupils see how $\frac{3}{4}$ can be changed to $\frac{15}{20}$:*

$$\frac{3}{4} = \frac{\boxed{5} \times 3}{\boxed{5} \times 4} = \frac{15}{20}$$

---

*Remind the pupils that the replacements for the two same-shaped frames in the same exercise must be the same number.

Ask the following questions:

(1) "By what number do we multiply the numerator $\frac{3}{\ }$ to get 15? 5

By what number do we multiply the denominator $\frac{\ }{4}$ to get 20?" 5

(2) "Do we change the value of $\frac{3}{4}$ when we multiply both terms (numerator and denominator) by the same number? No Why?" $\frac{5}{5}$ is equal to 1; therefore, multiplying a number by 1 does not change its value.

For individual or group practice, write the following examples on the chalkboard. Have the pupils copy each and put in the missing numerals to show how $\frac{3}{4}$ can be expressed by other equivalent fractions.

	(a)	(b)
(1)	$\frac{3}{4} = \frac{\boxed{2} \times 3}{\boxed{2} \times 4} = \frac{6}{8}$	$\frac{3}{4} = \frac{\boxed{3} \times 3}{\boxed{3} \times 4} = \frac{9}{12}$
*(2)	$\frac{3}{4} = \frac{\boxed{8} \times 3}{\boxed{8} \times 4} = \frac{24}{32}$	$\frac{3}{4} = \frac{\boxed{7} \times 3}{\boxed{7} \times 4} = \frac{21}{28}$

Next, have the pupils start with this pattern sentence: $\frac{3}{8} = \frac{\boxed{\ } \times 3}{\boxed{\ } \times 8} = \frac{?}{?}$.

Have them replace the $\boxed{\ }$ in the numerator and the denominator with 2; with 3; with 4; with 5. Ask, "What fractions do you get for each? $\frac{6}{16}$, $\frac{9}{24}$, $\frac{12}{32}$, and $\frac{15}{40}$ Do these fractions form a series of equivalent fractions for $\frac{3}{8}$?" yes

Pupil Exercises. Provide each pupil with worksheet No. 23.

Since many pupils will not be familiar with this type of notation, work some of the examples on the chalkboard before independent work is assigned.

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*If the pupils do not discover that the triangle-shaped frame in example 2 (a) should be replaced by 24, you may wish to ask, "By what numbers do we multiply 4 to get 32? Must both the denominator and the numerator of a fraction be multiplied by the same number to get an equivalent fraction?"

STUDY THE TWO SERIES OF EQUIVALENT FRACTIONS. WRITE FIVE MORE EQUIVALENT FRACTIONS WHICH CONTINUE THE SERIES IN (a) AND (b).

1. (a) $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \frac{7}{14}, \frac{8}{16}, \frac{9}{18}$

(b) $\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \frac{12}{18}, \frac{14}{21}, \frac{16}{24}$

WRITE A SERIES OF OUR EQUIVALENT FRACTIONS FOR EACH OF THE FOLLOWING:

2. (a) $\frac{1}{6} = \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \frac{5}{30}$

(b) $\frac{3}{7} = \frac{6}{14}, \frac{9}{21}, \frac{12}{28}, \frac{15}{35}$

IN EACH EXAMPLE, REPLACE THE FRAMES WITH THE MISSING NUMERALS.

3. (a) $\frac{2}{3} = \frac{\boxed{7} \times 2}{\boxed{7} \times 3} = \frac{14}{21}$

(b) $\frac{5}{6} = \frac{\boxed{7} \times 5}{\boxed{7} \times 6} = \frac{35}{42}$

4. (a) $\frac{3}{8} = \frac{\boxed{6} \times 3}{\boxed{6} \times 8} = \frac{18}{48}$

(b) $\frac{1}{3} = \frac{\boxed{5} \times 1}{\boxed{5} \times 3} = \frac{5}{15}$

5. (a) $\frac{2}{5} = \frac{\boxed{5} \times 2}{\boxed{5} \times 5} = \frac{\triangle 10}{25}$

(b) $\frac{5}{6} = \frac{\triangle 4 \times 5}{\triangledown 4 \times 6} = \frac{\boxed{20}}{24}$

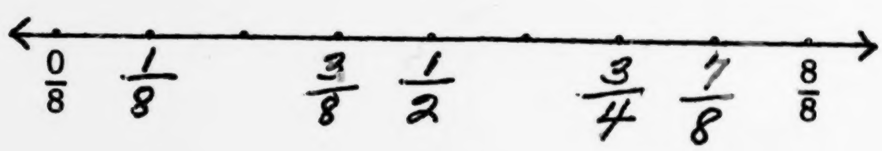
6. (a) $\frac{3}{7} = \frac{\boxed{4} \times 3}{\boxed{4} \times 7} = \frac{12}{\triangle 28}$

(b) $\frac{2}{3} = \frac{8 \times 2}{8 \times 3} = \frac{\triangle 16}{\boxed{24}}$

7. (a) $\frac{3}{4} = \frac{\triangle 27}{36}$ (b) $\frac{3}{8} = \frac{\triangle 9}{24}$ (c) $\frac{2}{3} = \frac{\triangle 14}{21}$ (d) $\frac{5}{6} = \frac{\triangle 25}{30}$

MATCH EACH OF THE FRACTIONS WITH A POINT ON THE NUMBER LINE (in order from least to greatest).

8. $\frac{3}{8}, \frac{1}{8}, \frac{3}{4}, \frac{1}{2}, \frac{7}{8}$



3. Using Equivalent Fractions

Objective. To introduce pupils to a method of finding the least common denominator when adding or subtracting fractional numbers.

Background Information for the Teacher. After studying equivalent fractions, pupils should not have difficulty with addition and subtraction of fractional numbers having unlike denominators. The basic structure which applies to addition of fractional numbers having like denominators applies also to fractional numbers having unlike denominators. The only new idea pupils should discover is that in addition and subtraction of fractional numbers the fractions naming the numbers must have like denominators. For example, consider the following:

$$\frac{1}{3} + \frac{1}{6} = \square$$

$$\begin{aligned} \frac{1}{3} + \frac{1}{6} &= \frac{2}{6} + \frac{1}{6} \\ &= \frac{2 + 1}{6} \\ &= \frac{3}{6} \end{aligned}$$

$$\left[\frac{1}{3} = \frac{2}{2} \times \frac{1}{3} = \frac{2}{6} \right]$$

Process for
an equivalent
fraction.

We know from our experience with the number line that $\frac{3}{6} = \frac{1}{2}$. Since we can multiply the numerator and the denominator by the same factor without changing the value of the fractional number, the inverse of this is also true. That is, the numerator and the denominator can be divided by the same factor (in this case by 3) without changing the value of the fractional number. For example, consider the following:

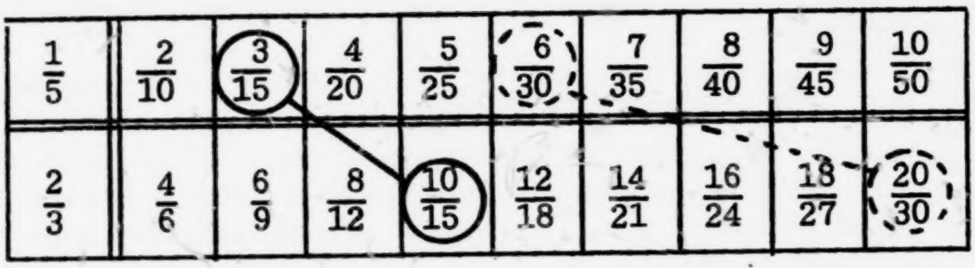
$$\frac{3}{6} = \frac{3}{6} \div \frac{3}{3} = \frac{3 \div 3}{6 \div 6} = \frac{1}{2}$$

Thus, before fractional numbers such as $\frac{1}{4}$ and $\frac{1}{3}$ can be added or subtracted, a common denominator must be found. In the case of $\frac{1}{4} + \frac{1}{3} = \square$, the common denominators are {12, 24, 36, 48, ... }; that is, any multiple of 12. However, it is usually convenient to use the least common denominator which would be 12. When the least common denominator is used, the answer is expressed in lower terms (not necessarily the lowest).

TEACHING PROCEDURE

Exploration. Work cooperatively with the class, eliciting answers. discussion, and questions when necessary. Make certain that the pupils understand how to change unlike fractions to like fractions. Writing the two columns of equivalent fractions (Figure 1) on the chalkboard may help to increase understanding.

Figure 1



$\frac{1}{5}$	$\frac{2}{10}$	$\frac{3}{15}$	$\frac{4}{20}$	$\frac{5}{25}$	$\frac{6}{30}$	$\frac{7}{35}$	$\frac{8}{40}$	$\frac{9}{45}$	$\frac{10}{50}$
$\frac{2}{3}$	$\frac{4}{6}$	$\frac{6}{9}$	$\frac{8}{12}$	$\frac{10}{15}$	$\frac{12}{18}$	$\frac{14}{21}$	$\frac{16}{24}$	$\frac{18}{27}$	$\frac{20}{30}$

Ask the following questions:

(1) "Before we can add $\frac{1}{5} + \frac{2}{3}$, what initial step is necessary?" We must find equivalent fractions for both $\frac{1}{5}$ and $\frac{2}{3}$ with common denominators.

(2) "In Figure 1, opposite the fraction $\frac{1}{5}$ is a series of equivalent fractions which are different names for $\frac{1}{5}$. Opposite the fraction $\frac{2}{3}$ there is another series of equivalent fractions which are different names for $\frac{2}{3}$. Select a fraction for $\frac{1}{5}$ and another for $\frac{2}{3}$ that will help us add $\frac{1}{5}$ and $\frac{2}{3}$. What is the common denominator for the two fractions $\frac{1}{5}$ and $\frac{2}{3}$? 15 Is it true that $\frac{1}{5} = \frac{3}{15}$? Yes That $\frac{2}{3} = \frac{10}{15}$? Yes Is this number sentence true? $\frac{1}{5} + \frac{2}{3} = \frac{3}{15} + \frac{10}{15}$? Yes What is the sum of $\frac{3}{15} + \frac{10}{15}$?" $\frac{13}{15}$

(3) "If we had chosen $\frac{2}{10}$ from the series (Figure 1) for $\frac{1}{5}$ and $\frac{6}{9}$ from the series for $\frac{2}{3}$, could we have found the sum? No Why?" The denominators are different.

(4) "Are there other equivalent fractions in Figure 1 that will help us add $\frac{1}{5} + \frac{2}{3} = \square$? Yes, $\frac{6}{30}$ and $\frac{20}{30}$ Why is it better to use $\frac{3}{15}$ and $\frac{10}{15}$?" The answer $(\frac{13}{15})$ is in lowest terms.

(5) "Pick two fractions from Figure 1 that will help with this number sentence: $\frac{2}{3} - \frac{1}{5} = \square$. $\frac{2}{3} = \frac{10}{15}$; $\frac{1}{5} = \frac{3}{15}$

Write a number sentence renaming $\frac{2}{3} - \frac{1}{5} = \square$ which would make subtraction easier because of common denominators. $\frac{10}{15} - \frac{3}{15} = \frac{7}{15}$

For individual practice, have the pupils copy the following examples from the chalkboard:*

	(a)	(b)	(c)
(1)	$\frac{3}{5} + \frac{1}{3} = \boxed{\frac{14}{15}}$	$\frac{1}{2} + \frac{1}{3} = \boxed{\frac{5}{6}}$	$\frac{4}{5} - \frac{1}{2} = \boxed{\frac{3}{10}}$
(2)	$\frac{3}{4} - \frac{1}{3} = \boxed{\frac{5}{12}}$	$\frac{7}{8} - \frac{2}{3} = \boxed{\frac{5}{24}}$	$\frac{1}{4} + \frac{2}{5} = \boxed{\frac{13}{20}}$

Next, introduce the pupils to the multiplication method of finding a common denominator. Illustrate the following series chart (Figure 2) on the chalkboard to find fractions equivalent to $\frac{3}{5}$ and $\frac{2}{7}$ that have a common denominator.

Figure 2

$\frac{3}{5}$	$\frac{6}{10}$	$\frac{9}{15}$	$\frac{12}{20}$	$\frac{15}{25}$	$\frac{18}{30}$	$\frac{21}{35}$
$\frac{2}{7}$	$\frac{4}{14}$	$\frac{6}{21}$	$\frac{8}{28}$	$\frac{10}{35}$	$\frac{12}{42}$	$\frac{14}{49}$

Ask the following questions:

(1) "What denominator from the chart is common to $\frac{3}{5}$ and $\frac{2}{7}$? 35 Name the equivalent fraction for $\frac{3}{5}$ with a denominator of 35. $\frac{21}{35}$ For $\frac{2}{7}$? $\frac{10}{35}$ Is this sentence true: $\frac{3}{5} + \frac{2}{7} = \frac{21}{35} + \frac{10}{35}$? Yes What is the sum of $\frac{3}{5} + \frac{2}{7}$?" $\frac{31}{35}$

(2) "Now let's try using multiplication to find equivalent like fractions. Study this example:

$$\frac{3}{5} = \frac{\boxed{7} \times 3}{\boxed{7} \times 5} = \frac{21}{35}$$

Can we use the denominator of $\frac{2}{7}$ to replace the two square-shaped frames? Yes Study this example:

$$\frac{2}{7} = \frac{\boxed{5} \times 2}{\boxed{5} \times 7} = \frac{10}{35}$$

Can we use the denominator of $\frac{3}{5}$ to replace the two square-shaped frame Yes.

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*Pupils may develop a series chart such as shown in Figure 1 to find a common denominator. Have the pupils show their work in the form: $\frac{3}{5} + \frac{1}{3} = \frac{9}{15} + \frac{5}{15} = \frac{14}{15}$.

(3) "What are the two answers to the examples in question (2)? $\frac{3}{5} = \frac{21}{35}$; $\frac{2}{7} = \frac{10}{35}$ What is the common denominator? 35 Are these the same two equivalent fractions we found in the series chart, Figure 2, when answering question (1)? Yes What is a shortcut method for finding a common denominator for two unlike fractions?* Multiply the two denominators.

Now, use a subtraction example and ask the pupils to find equivalent like fractions for $\frac{2}{3}$ and $\frac{1}{7}$ to complete this example: $\frac{2}{3} - \frac{1}{7} = \square$.

Ask the following questions:

(1) "Will the product of the denominators, 3 × 7, be a common denominator? Yes What is this common denominator?" 21

(2) "If we use the denominator of $\frac{1}{7}$ to replace the square-shaped frames, what will be the equivalent fraction for $\frac{2}{3}$?"

$$\frac{2}{3} = \frac{\boxed{7} \times 2}{\boxed{7} \times 3} = \frac{14}{21}$$

Now, use the denominator of $\frac{2}{3}$ to find an equivalent fraction for $\frac{1}{7}$:"

$$\frac{1}{7} = \frac{3 \times 1}{3 \times 7} = \frac{3}{21}$$

(3) "Is this sentence true: $\frac{2}{3} - \frac{1}{7} = \frac{14}{21} - \frac{3}{21}$? Yes Find the missing numeral for: $\frac{2}{3} - \frac{1}{7} = \frac{14}{21} - \frac{3}{21} = \square$." $\frac{11}{21}$

Write the following examples on the chalkboard and work several cooperatively with the class. Be sure to work at least one addition and subtraction sentence involving mixed fractions.

	(a)	(b)	(c)
(1)	$\frac{2}{3} - \frac{1}{2} = \boxed{\frac{1}{6}}$	$\frac{5}{8} + \frac{1}{3} = \boxed{\frac{23}{24}}$	$3\frac{1}{3} + 4\frac{1}{4} = \boxed{7\frac{7}{12}}$
(2)	$6\frac{1}{3} - 2\frac{1}{8} = \boxed{4\frac{5}{24}}$	$5\frac{2}{3} + 2\frac{1}{4} = \boxed{7\frac{11}{12}}$	$8\frac{1}{3} + 6\frac{3}{5} = \boxed{14\frac{14}{15}}$

---

*Point out to the pupils that this method can always be used to find a common denominator. However, this method does not always produce the least common denominator.

Before introducing the pupils to the concept of the least common denominator, ask, "What is a common denominator for the fractions $\frac{5}{6}$ and $\frac{3}{8}$? 48 Express $\frac{5}{6}$ and $\frac{3}{8}$ as like equivalent fractions.

$$\frac{5}{6} + \frac{3}{8} = \frac{40}{48} + \frac{18}{48} \quad \frac{\boxed{58}}{\triangle\,48}$$

Find the sum in lowest terms. $58 \div 48 = 1\frac{5}{24}$

Next, have the pupils study the series chart (Figure 3) for $\frac{5}{6}$ and $\frac{3}{8}$.

Figure 3

$\frac{5}{6}$	$\frac{3}{8}$
$\frac{15}{18}$	$\frac{9}{24}$
$\frac{20}{24}$	$\frac{12}{32}$
$\frac{25}{30}$	$\frac{15}{40}$
$\frac{30}{36}$	$\frac{18}{48}$
$\frac{35}{42}$	$\frac{21}{56}$
$\frac{40}{48}$	$\frac{24}{64}$

Ask the following questions:

(1) "Can we find a pair of fractions having a common denominator less than 48? Yes What is it? 24 Express $\frac{5}{6}$ and $\frac{3}{8}$ as like equivalent fractions using 24 as the common denominator.

$$\frac{5}{6} + \frac{3}{8} = \frac{20}{24} + \frac{9}{24} = \frac{\boxed{29}}{\triangle\,24}$$

Find the sum in lowest terms."

$29 \div 24 = 1\frac{5}{24}$

(2) "Do we get the same answer when we use the least common denominator or a denominator which is greater than the least common denominator?"

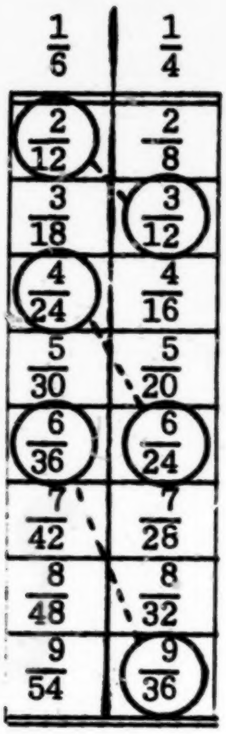
Yes

Point out to the pupils that there are an unlimited number of common denominators for two fractions, but only one least common denominator. They should be directed to see that the least common denominator is the most convenient denominator to use; its use results in the least work. This generalization should be understood rather than memorized.

An interesting exercise is to ask pupils to find examples where this method does not produce the least common denominator. These are compared with examples of those that do produce the least common denominators. Pupils analyze and discover the number property involved.

Use the following series chart (Figure 4) to illustrate that three pairs of like fractions can be used for adding $\frac{1}{6}$ and $\frac{1}{4}$; each will yield the same sum.

Figure 4



$\frac{1}{6}$	$\frac{1}{4}$
$\frac{2}{12}$	$\frac{2}{8}$
$\frac{3}{18}$	$\frac{3}{12}$
$\frac{4}{24}$	$\frac{4}{16}$
$\frac{5}{30}$	$\frac{5}{20}$
$\frac{6}{36}$	$\frac{6}{24}$
$\frac{7}{42}$	$\frac{7}{28}$
$\frac{8}{48}$	$\frac{8}{32}$
$\frac{9}{54}$	$\frac{9}{36}$

Ask the following questions:

(1) "Name three pairs of like equivalent fractions for $\frac{1}{6}$ and $\frac{1}{4}$:

(a) $\frac{1}{6} + \frac{1}{4} = \frac{2}{12} + \frac{3}{12}$

(b) $\frac{1}{6} + \frac{1}{4} = \frac{4}{24} + \frac{6}{24}$

(c) $\frac{1}{6} + \frac{1}{4} = \frac{6}{36} + \frac{9}{36}$

What is the sum for (a)? $\frac{5}{12}$

For (b)? $\frac{10}{24}$ or $\frac{5}{12}$

For (c)?" $\frac{15}{36}$ or $\frac{5}{12}$

(2) "What can we say about the sum of each pair of like equivalent fractions? The sum is the same; $\frac{5}{12}$. Which pair has the least common denominator?" (a); $\frac{1}{6} + \frac{1}{4} = \frac{2}{12} + \frac{3}{12}$

Now, introduce the pupils to a method which uses multiples of the largest denominator to find the least common denominator. Remind the pupils that unlike fractions must be changed to like fractions before they can be used in adding or subtracting. Also, point out that the denominators of the unlike fractions are factors of the common denominator.

Before proceeding with this method, review briefly the idea of multiples of a number. For example, the multiples of 5 are: 5, 10, 15, 20, etc.; multiples of 7 are: 7, 14, 21, 28, etc.; multiples of 10 are: 10, 20, 30, 40, etc.

Now ask the pupils to study the following example and find the least common denominator without the aid of a series chart:

$$\frac{7}{10} + \frac{1}{8} = \square$$

Ask the following questions:

(1) "Name several common denominators for $\frac{7}{10}$ and $\frac{1}{8}$. 40, 80, 160, 320, etc.

What is the least common denominator for $\frac{7}{10}$ and $\frac{1}{8}$?" 40

(2) "List the multiples of the larger of the two denominators, 10 and 8. 10, 20, 30, 40, 50, 60, 70, 80, etc. Is 10 a factor* of 10? yes Of 20? yes Of 30? yes Of 40? yes Of 50? yes, etc."

(3) "Is 8 a factor of 10? No Of 20? No Of 30? No Of 40? yes. Since both the denominators, 8 and 10, are factors of 40, but no number smaller than 40, is 40 the least common denominator?" yes

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* See GLOSSARY, page 179.

Now, write the following examples on the chalkboard and ask the pupils to write the equivalent fractions for $\frac{7}{10}$ and $\frac{1}{8}$:*

(a)

$$\frac{7}{10} = \frac{\boxed{4} \times 7}{\boxed{4} \times 10} = \frac{28}{40}$$

(b)

$$\frac{1}{8} = \frac{\boxed{5} \times 1}{\boxed{5} \times 8} = \frac{5}{40}$$

(4) "What is the sum of $\frac{7}{10} + \frac{1}{8}$? $\frac{7}{10} + \frac{1}{8} = \frac{28}{40} + \frac{5}{40} = \frac{33}{40}$

What is the difference of $\frac{7}{10} - \frac{1}{8}$?" $\frac{7}{10} - \frac{1}{8} = \frac{28}{40} - \frac{5}{40} = \frac{23}{40}$

Use the same procedure to find the sum of $\frac{5}{12} + \frac{3}{16} = \boxed{\frac{29}{48}}$ and the difference of $\frac{5}{12} - \frac{3}{16} = \boxed{\frac{11}{48}}$. Since this method is quite detailed, work several example cooperatively with the pupils. After considerable group work, have the pupils do the following for individual practice:

(a)

(1) $\frac{3}{8} + \frac{5}{12} = \frac{9}{24} + \frac{10}{24} = \frac{19}{24}$

(2) $\frac{5}{8} - \frac{1}{3} = \frac{15}{24} - \frac{\boxed{8}}{24} = \frac{7}{24}$

(b)

$\frac{1}{6} + \frac{3}{4} = \frac{2}{\boxed{12}} + \frac{9}{12} = \frac{11}{12}$

$\frac{11}{12} - \frac{3}{10} = \frac{55}{\boxed{60}} - \frac{18}{\boxed{60}} = \frac{3}{6}$

Pupil Exercises. Provide each pupil with worksheet No. 24.

For examples in 3 and 4, use the first members of each set to determ whether the other members of each set are equivalent.

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*Record the correct responses.

Interim Mathematics Unit - Grade 6 Worksheet No. 24
San Diego City Schools (1-65)

WRITE A NUMBER SENTENCE FOR EACH PROBLEM AND SOLVE. GIVE YOUR ANSWER IN LOWEST TERMS.

1. A portable water fountain contained $5\frac{1}{2}$ gallons of water. If $2\frac{1}{8}$ gallons were removed, how many gallons would remain? $5\frac{1}{2} - 2\frac{1}{8} = 3\frac{3}{8}$ gallons

2. A board $\frac{3}{8}$ inch thick is placed on top of a board that is $\frac{5}{16}$ inch thick. What is the total thickness of the boards? $\frac{3}{8} + \frac{5}{16} = \frac{11}{16}$ inch

IN EACH SET, CIRCLE THE FRACTION OR FRACTIONS WHICH IS (ARE) NOT EQUIVALENT TO THE OTHERS.

(a) (b) (c)

3. A = { $\frac{2}{3}$, $\frac{10}{15}$, $\frac{8}{12}$, [✓ $\frac{6}{4}$] } B = { $\frac{3}{8}$, $\frac{6}{16}$, [✓ $\frac{4}{6}$], $\frac{9}{24}$ } C = { $\frac{12}{16}$, $\frac{9}{12}$, $\frac{6}{8}$, $\frac{3}{4}$, [✓ $\frac{6}{16}$] }

4. D = { $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, [✓ $\frac{4}{6}$], [✓ $\frac{5}{9}$], $\frac{6}{12}$ } E = { $\frac{2}{3}$, [✓ $\frac{3}{6}$], $\frac{4}{6}$, [✓ $\frac{5}{8}$], [✓ $\frac{7}{10}$] }

REWRITE EACH OF THE FOLLOWING EXPRESSIONS SO THAT THE FRACTIONS HAVE A COMMON DENOMINATOR. THEN ADD OR SUBTRACT. EXPRESS ANSWERS IN LOWEST TERMS.

(a) (b)

5. $\frac{1}{2} + \frac{1}{6} = \frac{3}{6} + \frac{1}{6} = \frac{4}{6}$ or $\frac{2}{3}$ $\frac{2}{3} + \frac{1}{6} = \frac{4}{6} + \frac{1}{6} = \frac{5}{6}$

6. $\frac{7}{8} - \frac{1}{4} = \frac{7}{8} - \frac{2}{8} = \frac{5}{8}$ $\frac{3}{4} - \frac{1}{6} = \frac{9}{12} - \frac{2}{12} = \frac{7}{12}$

COMPLETE THE FOLLOWING SEQUENCES:

7. (a) $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3. (b) $1\frac{1}{2}$, 3, $4\frac{1}{2}$, 6, $7\frac{1}{2}$, 9

(c) 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$. (d) $\frac{1}{3}$, 1, $1\frac{2}{3}$, $2\frac{1}{3}$, [illegible], $3\frac{2}{3}$

COMPLETE EACH OF THE FOLLOWING WHICH ARE NAMES FOR WHOLE NUMBERS.

8. (a) $\frac{12}{2}$ = 6 (b) $\frac{71}{11}$ = (c) $\frac{3}{4}$ =

(d) $\frac{22}{22}$ = 1 (e) $1\frac{2}{4}$ = (f) $\frac{12}{6}$ = 2

B. Decimals*

1. Concept of Decimals

Objective. To develop an appreciation and understanding of decimals as part of the numeral system.

Background Information for the Teacher. The decimal language has certain advantages for computation which the fractional number language does not have. Therefore it is important that the pupils learn about numbers named by decimals. For example, if we start with decimal names for whole numbers such as 345, we see that

345 means: 3 hundreds + 4 tens and 5 ones

or 34 tens and 5 ones

or 3 hundreds + 45 ones, etc.

Similarly,

3.45 means: 3 ones + 4 tenths + 5 hundredths

or 34 tenths + 5 hundredths

or 2 ones + 14 tenths + 5 hundredths, etc.

To use another example, the following number sentence will illustrate a special case of the use of decimal notation:

$$3.672 = 3 + \frac{6}{10} + \frac{7}{100} + \frac{2}{1000}$$

In order to read 3.672 as "three and** six hundred seventy-two thousandths," we must show that (given the basic definition of 3.672 as shown above):

$$3 + \frac{6}{10} + \frac{7}{100} + \frac{2}{1000} = 3\frac{672}{1000}$$

For the purpose of clarity and precision in the use of terms, the following are defined:

---

*The word decimal will be used synonymously with decimal fraction.

** Emphasize the fact that when saying names of fractional numbers greater than 1, the word "and" is used to separate the whole number name and the fractional number name. Thus, $36\frac{1}{2}$ is said "thirty-six and one half"; 45.7 is said "forty-five and seven tenths."

(a) .7 This numeral is called a decimal fraction or decimal. The dot in front of the 7 is called a decimal point. We do not write the denominator when we use the decimal point to write a fraction.

(b) $\frac{7}{10}$ This numeral is called a common fraction or fraction.

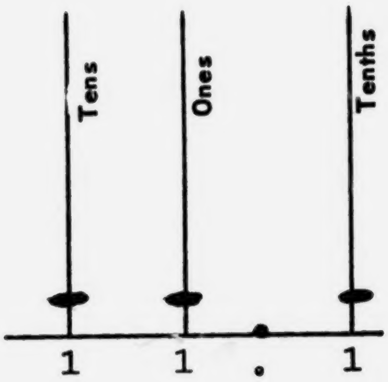
(c) 13.4 This numeral is sometimes called a decimal mixed fraction. However, most frequently decimal fractions and decimal mixed fractions are called decimals.

(d) $11\frac{3}{10}$ This numeral is called a mixed fraction. It includes the whole number name and the common fraction.

TEACHING PROCEDURE

Exploration. Copy or illustrate the following chart (Figure 1) on the chalkboard to represent the number 11.1. Ask these or similar questions:

Figure 1

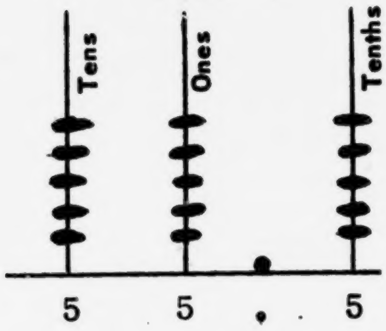


(1) "Is the value of the 1 in the tens' place ten times the value of the 1 in the ones' place? Yes Can we say that 1 is $\frac{1}{10}$ of 10?" Yes

(2) "Since the fraction $\frac{10}{10}$ is another name for 1, is the 1 in the ones' place 10 times the value as the 1 in the tenths' place? Yes Is the 1 in the tenths' place $\frac{1}{10}$ of 1?" Yes

Next, illustrate the chart (Figure 2) as shown, with the digit 5 in each place.

Figure 2



Ask the following questions:

(1) "The 5 in the tens' place is how many times the value of 5 in the one's place? 10 times The 5 in the one's place is how many times the value of the 5 in the tenths' place?" 10 times

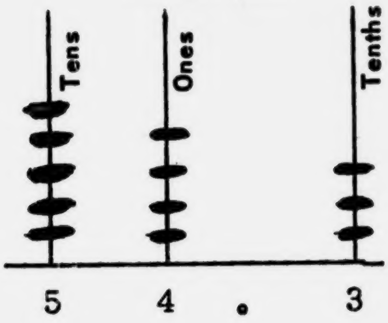
(2) "The 5 in the tenths' place is what part of the value of the 5 in the ones' place?" $\frac{1}{10}$ times

For further development and understanding of place-value, write some of the following examples on the chalkboard as an oral activity.*

	(a)	(b)
(1)	1 ten = 10 ones	10 ones = 1 ten
(2)	4 tens = 40 ones	40 ones = 4 tens
(3)	20 tenths = 2 ones	3 ones = 30 tenths
(4)	40 tenths = 4 ones	8 ones = 80 tenths
(5)	100 tenths = 10 ones	10 ones = 100 tenths

Illustrate Figure 3 on the chalkboard (or use a place-value chart) and ask the following questions:

Figure 3



(1) "What decimal mixed fraction is shown in the place-value chart?" 54.3

(2) "How many tens, ones, and tenths does the numeral 54.3 represent?" 5 tens, 4 ones, and 3 tenths

(3) "How many ones and tenths does the numeral 54.3 represent?" 54 ones and 3 tenths

For individual practice, have the pupils copy (from the chalkboard) and write in the missing numeral for each of the following examples:

	(a)	(b)
(1)	.8 = 8 tenths	3.0 = 30 tenths
(2)	.6 = 6 tenths	1.8 = 18 tenths
(3)	1.0 = 10 tenths	63.6 = 636 tenths
(4)	3.6 = 36 tenths	87.4 = 874 tenths
(5)	7.4 = 74 tenths	30.0 = 300 tenths

---

*After each correct response, record the correct numeral in each blank space.

2. The Decimal Concept of Hundredths

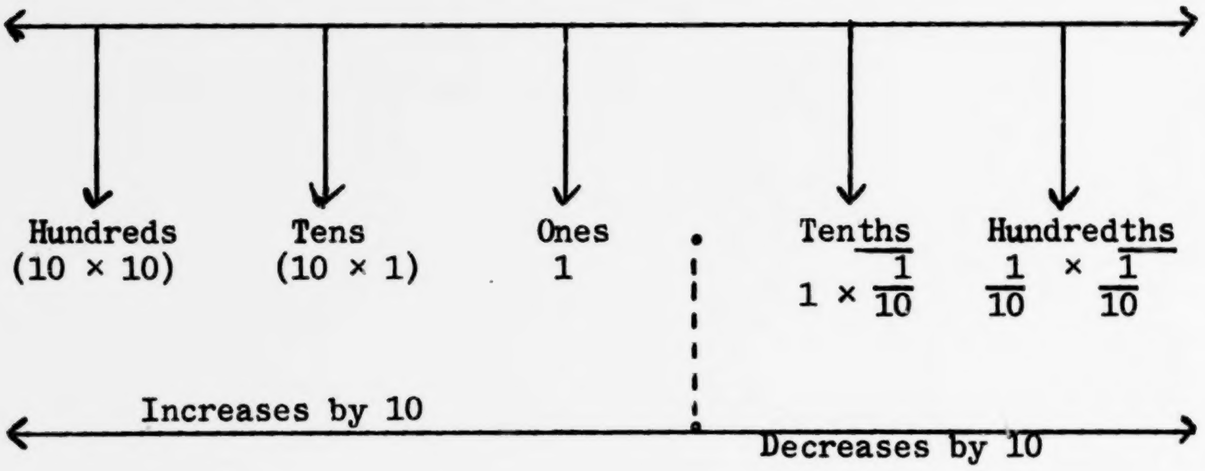
Objective. To provide pupils with experiences which will lead to the discovery and understanding of place-value in decimals.

Background Information for the Teacher. The pupils should become aware of and understand the position of the ones' place in a decimal numeral through the use of a place-value chart or other device which will show (a) that the tens' place is one place to the left of the ones' place and that the tenths' place is one place to the right of ones' place; (b) hundreds' place is two places to the left of the ones' place and the hundredths' place is two places to the right of the ones' place. A similar awareness should be made when extending the concept to thousands and thousandths.

The pupils should see that the value of each successive position in our number system increases as we move from right to left, always by a multiplication of 10. As we move from left to right, the value of each successive position decreases by a divisor of 10.

To extend the meaning of place-value and to show the sequence of decimal places, use the chart shown below.

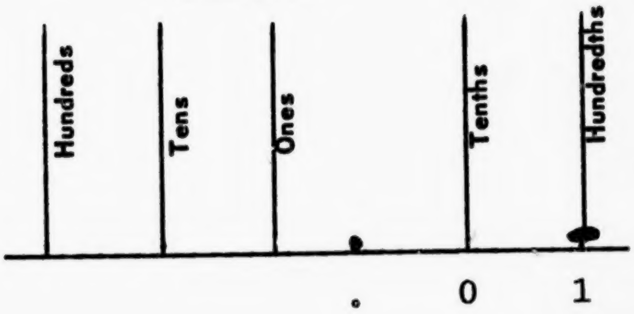
Decimal Notation



TEACHING PROCEDURE

Exploration. Use the following illustration (Figure 1) of a place-value chart to show that the hundreds' place is two places to the left of ones' place and that the hundredths' place is two places to the right of ones' place. Ask the following:

Figure 1



(1) "What does the digit in the hundredths' place represent?"

1 hundredth

(2) "What fractional numeral can we write to express .01?" $\frac{1}{100}$

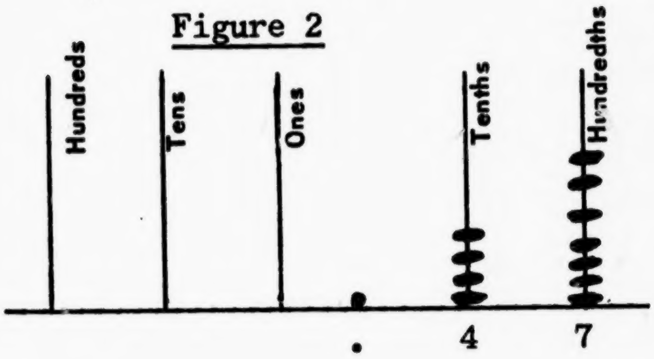
(3) "How do we read .01 and $\frac{1}{100}$?"

Both are read the same, "one hundredth"

In the same place-value chart, write the digit 2 in the hundredths' place. Ask, "Does the 2 mean $\frac{2}{100}$? Yes How do we write the decimal that means 2 hundredths?" .02

Next, illustrate the following chart (Figure 2) with the digit 4 in in tenths' place and the digit 7 in the hundredths' place. As the following:

Figure 2



(1) "What digit is in the hundredths' place? 7 What digit is in the tenths' place? 4 How is the decimal shown? .47 How do we read this numeral? forty-seven hundredths Write the decimal .47 as a fraction." $\frac{47}{100}$

(2) "Are $\frac{47}{100}$ and .47 names for 47 hundredths? Yes Is it true that $.47 = \frac{47}{100}$?" Yes

For group or individual practice, have the pupils copy and do the following examples. Write these on the chalkboard.

Write a decimal for each of the following:

	(a)	(b)
(1)	$\frac{5}{100}$ = .05	$\frac{17}{100}$ = .17
(2)	$\frac{9}{100}$ = .09	$\frac{58}{100}$ = .58
(3)	$\frac{7}{100}$ = .07	$\frac{95}{100}$ = .95

Write in the missing numerals.

(1) .07, .08, .09, .10, .11, .12, .13.

(2) .04, .06, .08, .10, .12, .14, .16.

(3) .33, .30, .27, .24, .21, .18, .15.

Pupil Exercises. Provide each pupil with worksheet No. 25.

For the members of the more able group who have a working knowledge of fractions and decimals, a minimum of explanation will be necessary. Others may need supervision and additional explanation.

WRITE A WORD NAME FOR EACH EXAMPLE.

1. .19 is called a decimal fraction or decimal.

2. $\frac{7}{16}$ is called a common fraction or fraction.

3. 15.07 is called a decimal mixed fraction or decimal.

4. $11\frac{5}{8}$ is called a mixed fraction.

WRITE A DECIMAL FOR EACH OF THE FOLLOWING:

5. (a) 7 tens + 5 ones + 6 tenths 75.6

(b) 35 ones + 9 tenths 35.9

(c) 38 tenths 3.8

(d) 40 tenths 4.0

(e) 76 tenths 7.6

WRITE THE MISSING WORD AND NUMERAL FOR EACH EXAMPLE. THE FIRST ONE IS DONE FOR YOU.*

6. (a) 28.45 The 2 in the tens' place means 20.

(b) 28.45 The 4 in the tenths' place means $\frac{4}{10}$ or .4

(c) 31.64 The 4 in the hundredths' place means $\frac{4}{100}$ or .04

(d) .32 The 2 in the hundredths' place means $\frac{2}{100}$ or .02

(e) 46.37 The 6 in the ones place means 6

FOR EACH EXAMPLE, WRITE A MIXED FRACTION AND A DECIMAL.

7. (a) 8 tens, 3 ones, and 9 tenths $83\frac{9}{10}$ 83.9

(b) 1 ten, 4 ones, and 3 tenths $14\frac{3}{10}$ 14.3

(c) 15 tens, 8 ones, and 8 hundredths $158\frac{8}{100}$ 158.08

(d) 87 ones and 19 hundredths $87\frac{19}{100}$ 87.19

WRITE THE MISSING NUMERAL IN EACH BLANK.

8. (a) .07, .09, .11, .13, .15, .17, .19, .21, .23, .25

(b) 1.11, 1.22, 1.33, 1.44, 1.55, 1.66, 1.77, 1.88, 1.99,

(c) 7.75, 8.75, 9.75, 10.75, 11.75, 12.75, 13.75, 14.75, 15.75

---

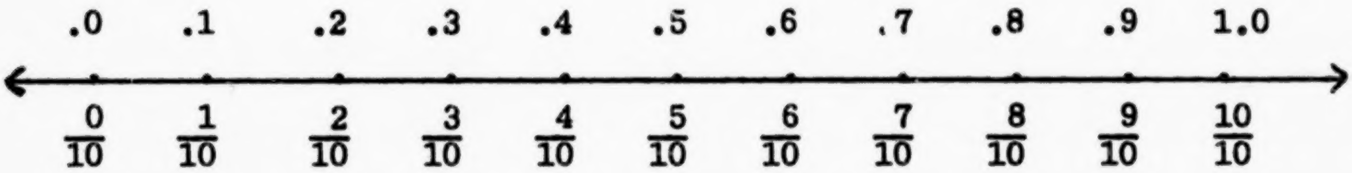
*Either form of the numeral is acceptable, that is, fraction or decimal.

3. Adding and Subtracting Decimals

Objective. To emphasize the relationship between addition and subtraction of whole numbers, like fractions, and decimals.

Background Information for the Teacher. In adding and subtracting decimals, pupils should be encouraged to apply the basic principles and techniques used with whole numbers and fractions. For example, they should relate the place-value system to fractional representations on the number line as shown below (Figure 1).

Figure 1



Expanded notation can also be used to develop and reinforce the meaning of numerals written to the right of the decimal point, thus making computation more meaningful. For example, we often use the vertical form (a) for adding mixed fractions. We can also use the vertical form (b) for adding decimals.

(a)

$$\begin{array}{rl} 2\frac{7}{10} & = 2 + \frac{7}{10} \\ +\; \underline{3\frac{1}{10}} & = \underline{3 + \frac{1}{10}} \\ & = 5 + \frac{8}{10} \\ & = 5\frac{8}{10} \end{array}$$

(b)

$$\begin{array}{rl} 2.7 & = 2 + .7 \\ +\; \underline{3.1} & = \underline{3 + .1} \\ & = 5 + .8 \\ & = 5.8 \end{array}$$

If the pupils understand place-value, they should discover that in addition and subtraction of decimals, the decimal point is always aligned and that numerals must be expressed with their proper place-value. In the example below, a column addition is shown.

$$\begin{array}{r} .23 \\ .17 \\ +\; \underline{.11} \\ .51 \end{array}$$

$$1 \times \frac{1}{100} = .01$$

$$5 \times \frac{1}{10} = .5$$

$$\overline{.51}$$

TEACHING PROCEDURE

Exploration. Introduction to this section may be made by way of the following questions: "Is .6 another name for $\frac{6}{10}$? Yes Is .3 another name for $\frac{3}{10}$? Yes Is the following sentence true? $.6 + .3 = \frac{6}{10} + \frac{3}{10}$? Yes What is the missing decimal in this sentence: $.6 + .3 = N$?" N = .9

To review another form of notation, ask, "Does .8 + .5 equal $\frac{8}{10} + \frac{5}{10}$? Yes Is it true that $\frac{8}{10} + \frac{5}{10}$ and $\frac{8 + 5}{10}$ are two different names for .8 + .5? Yes What is the decimal for $\frac{13}{10}$ or $1\frac{3}{10}$? 1.3 Then, what decimal will replace N in this sentence: $.8 + .5 = N$?" 1.3

For individual practice, write the following examples on the chalkboard. Have the pupils replace N with the missing decimal. If necessary, have the pupils use the pattern for addition with like fractions to check their work.

	(a)	(b)
(1)	.6 + .8 = N 1.4	.7 + .7 = N 1.4
(2)	.9 + .8 = N 1.7	.5 + .5 = N 1.0 or 1
(3)	.4 + .5 = N .9	.2 + .9 = N 1.1

To extend addition to hundredths and thousandths, indicate that the point is placed in the proper position only after the addition has been completed. Thus, thinking about the decimal point may be postponed and need not interfere with the computation. In other words, add just as you would with whole numbers and place the decimal point in the proper position. Lead the pupils into addition of decimals by way of the following questions:

(1) "Does .54 + .72 equal $\frac{54}{100} + \frac{72}{100}$? Yes What is the sum of $\frac{54 + 72}{100}$?" $\frac{126}{100}$

(2) "What is the decimal name for $\frac{126}{100}$? 1.26 If we add .54 and .72, what is the sum?" 1.26

Write the following examples on the chalkboard for individual practice. Have the pupils replace N with the missing numeral. Encourage the use of the method used in question 2 above.

	(a)	(b)
(1)	.63 + .26 = N .89	2.31 + .42 = N 2.73
(2)	.52 + .66 = N 1.18	4.54 + .63 = N 5.17
(3)	.85 + .94 = N 1.79	8.06 + .57 = N 8.63

Next, extend the addition of decimals by emphasizing that the same procedure used for adding with numerals for whole numbers may be applied to addition with like decimals. Use the following examples (a) and (b) below:

(a)

$$\begin{array}{rcl} 5.46 & = & 5 + .4 + .06 \\ + \underline{7.32} & = & \underline{7 + .3 + .02} \\ & = & 12 + .7 + .08 \\ & = & 12.78 \end{array}$$

(b)

$$\begin{array}{r} 5.46 \\ + \underline{7.32} \\ 12.78 \end{array}$$

Ask the following questions:

(1) "In example (a), what form is used to illustrate addition of decimals?"

Expanded notation

(2) "In example (b), what form is used to illustrate addition of decimals?"

Vertical notation

(3) "What general statement can we make about the two forms when adding with decimals?"

You can add with decimals just as you add with numerals for whole numbers. This is followed by placing the decimal point in the answer to indicate the correct place-value of each digit.

It is often necessary to regroup when adding with decimals. This development parallels the regrouping process of whole numbers. If the pupils understand place-value and the regrouping process, they will readily see the pattern and relationships.

Write the following example on the chalkboard and have the pupils tell what the missing numerals are. Record the pupils' responses in the blank spaces.

$$\begin{array}{rl} 3.68 = & 3 \text{ ones, } 6 \text{ tenths, } 8 \text{ hundredths} \\ + \underline{9.79} = & \underline{9 \text{ ones, } 7 \text{ tenths, } 9 \text{ hundredths}} \\ = & 12 \text{ ones, } 13 \text{ tenths, } 17 \text{ hundredths} \\ = & 13 \text{ ones, } 4 \text{ tenths, } 7 \text{ hundredths} \\ = & 13.47 \end{array}$$

Ask the following questions:

(1) "If we regroup the 17 hundredths into tenths and hundredths, what should we do with the 1 tenth?" Add it to the 13 tenths

(2) "If we regroup the 14 tenths (13 tenths plus 1 tenth) into ones and tenths, how many ones do we add to the 12 ones? 1 What is the sum of 3.68 + 9.78 = N?" 13.47

Next, write the following example on the chalkboard to show that there is another way to find the sum of 3.68 and 9.79.

```
   3.68
 + 9.79
 ------
    .17
   1.3
  12.
 ------
  13.47
```

Ask the following questions:

(1) "What is the sum of 8 hundredths and nine hundredths? 17 hundredths
Where does the 1.3 come from? the sum of 6 tenths and 7 tenths
What is the sum of 3 ones + 9 ones?" 12

(2) "If we add the three partial sums (.17 + 1.3 + 12), what is the solution or answer?" 13.47

Another way of showing the addition of 3.68 and 9.79 is by way of the following example:

```
   3.68
 + 9.79
 ------
  13.47
```

Ask the following questions:

(1) "When we add the hundredths and regroup the sum, what do we write in the hundredths place?" 7

(2) "Remembering the 1 tenth, we add the tenths. What is the sum of 1 tenth, 6 tenths, and 7 tenths? 14 tenths What numeral is written in the tenths' place in the sum? 4 In adding the ones what is the sum? 13 ones What decimal do we write as the sum of 3.68 and 9.79? 13.47 Is the sum the same as that shown in the two previous examples?" Yes

Have the pupils copy the following examples from the chalkboard for individual practice:

	(a)	(b)	(c)	(d)
(1)	36.8 + 27.6 64.4	.89 + .65 1.54	16.04 + 9.57 25.61	9.34 + 2.69 12.03

	(a)	(b)	(c)	(d)
(2)	61.09	7.88	13.72	6.54
	28.34	9.63	9.67	9.74
	+ 19.73	+ 4.05	+ 7.35	5.86
	109.16	21.56	30.74	22.14

Introduction to subtraction may be made by way of the following questions: "Is 5.69 - 3.24 another name for $\frac{569 - 324}{100}$? Yes If we subtract 324 from 569, what is the difference? 245 What is the decimal for $\frac{245}{100}$? 2.45 Then, what decimal should we use to replace N in this sentence: 5.69 - 3.24 = N?" 2.45

Indicate to the pupils that the vertical form can also be used in subtracting with decimals. Have them examine examples (a) and (b) below:*

(a)

7.68 = 7 ones, 6 tenths, 8 hundredths
− 3.24 = 3 ones, 2 tenths, 4 hundredths
= 4 ones, 4 tenths, 4 hundredths
= 4.44

(b)

$$\begin{array}{r} 7.68 \\ -\ 3.24 \\ \hline 4.44 \end{array}$$

Ask the following questions:

(1) "In example (a), is regrouping necessary? No What form is used to illustrate subtraction of decimals?" Expanded form

(2) "In example (b), what form is used to illustrate subtraction of decimals?" Vertical form

(3) "In example (a) and (b), where is the decimal point located?" Between ones' place and tenths' place.

(4) "What general statement can we make about the two forms when subtracting with decimals?" You can subtract with decimals just as you subtract with numerals for whole numbers. This is followed by placing the decimal point in the answer to indicate the place-value of each digit.

Write the following on the chalkboard for individual practice. Have the pupils write the missing decimals for N in each example:

---

*Record the correct responses given by the pupils.

	(a)	(b)	(c)
(1)	3.7 - .7 = N 3.0	4.86 - 2.32 = N 2.54	.63 - .20 = N .43
(2)	.76 - .04 = N .72	.83 - .40 = N .43	11.85 - 6.31 = N 5.54
(3)	.28 - .12 = N .16	.9 - .30 = N .6	2.66 - .42 = N 2.24
(4)	10.67 - 4.43 6.24	59.86 - 46.80 13.06	189.76 - 27.41 162.35

Next, indicate to the pupils that it is often necessary to regroup when subtracting with decimals. Have them study examples (a), (b), and (c) below to see the steps necessary to subtract 1.46 from 4.23.

(a)

4.23 = 4 ones, 2 tenths, 3 hundredths
- 1.46 = 1 one, 4 tenths, 6 hundredths

(b)

4.23 = 4 ones, 1 tenth, 13 hundredths
- 1.46 = 1 one, 4 tenths, 6 hundredths
7 hundredths

(c)

4.23 = 3 ones, 11 tenths, 13 hundredths
- 1.46 = 1 one, 4 tenths, 6 hundredths
= 2 ones, 7 tenths, 7 hundredths
= 2.77

Ask the following questions:

(1) "In example (a), can we subtract the hundredths without regrouping?" No

(2) "Is it possible to subtract hundredths in example (b) now that regrouping has taken place? Yes Can we subtract tenths? No

(3) "With regrouping completed in example (c), is it possible to complete the subtraction operation? Yes Name the decimal which completes the sentence 4.23 - 1.46 = N." 2.77

Show the vertical form which illustrates the mental process of regrouping which is necessary. Use example (d) as shown.

(d)

4.23
- 1.46

Ask the following questions:

(1) Does the vertical form provide a 'shortcut' method for subtracting? Yes What mental process of regrouping is necessary in this example?" First, we must think of "6 from 13 is 7 (hundredths)." Then, "4 from 11 is 7 (tenths), and 1 from 3 is 2 (ones.

(2) "Is the difference in example (c) and example (d) the same? Yes Name the decimal for 4.23 - 1.46 = N." 2.77

For individual practice, write the following examples on the chalkboard. Have the pupils write the missing decimals for each.

	(a)	(b)	(c)	(d)
(1)	$\begin{array}{r} 35.2 \\ -\ 26.8 \\ \hline 8.4 \end{array}$	$\begin{array}{r} 4.53 \\ -\ 3.77 \\ \hline .76 \end{array}$	$\begin{array}{r} 43.57 \\ -\ 26.87 \\ \hline 16.70 \end{array}$	$\begin{array}{r} 40.45 \\ -\ 3.58 \\ \hline 36.87 \end{array}$

	(a)	(b)
(2)	36.2 - 13.9 = N 22.3	62.9 + N = 92.8 29.9
(3)	5.34 - N = 4.68 .66	9.86 - N = 4.82 5.04

Pupil Exercises. Provide each pupil with worksheet No. 26.

Encourage independent work on the part of the pupils who understand decimals. Work cooperatively with the less able group.

Use the following approach to problem 6:

(a)

$$\begin{array}{lr} \text{One } (2 \times 2) \text{ square} & = 1 \\ \text{Four } (1 \times 1) \text{ squares} & = \underline{4} \\ & 5 \end{array}$$

(b)

$$\begin{array}{lr} \text{One } (3 \times 3) \text{ square} & = 1 \\ \text{Four } (2 \times 2) \text{ squares} & = 4 \\ \text{Nine } (1 \times 1) \text{ squares} & = \underline{9} \\ & 14 \end{array}$$

(c)

$$\begin{array}{lr} \text{One } (4 \times 4) \text{ square} & = 1 \\ \text{Four } (3 \times 3) \text{ squares} & = 4 \\ \text{Nine } (2 \times 2) \text{ squares} & = 9 \\ \text{Sixteen } (1 \times 1) \text{ squares} & = \underline{16} \\ & 30 \end{array}$$

Interim Mathematics Unit - Grade 6
San Diego City Schools (1-65)

Worksheet No. 26

RENAME EACH OF THE FOLLOWING AS MIXED FRACTIONS:

1. (a) 0.17 = $\frac{17}{100}$
   (b) 3.02 = $3\frac{2}{100}$
   (c) 5.19 = $5\frac{19}{100}$
   (d) 22.22 = $22\frac{22}{100}$
   (e) 1.09 = $1\frac{9}{100}$
   (f) .12 = $\frac{12}{100}$

RENAME EACH OF THE FOLLOWING AS A DECIMAL:

2. (a) $\frac{15}{100}$ = .15
   (b) $\frac{37}{100}$ = .37
   (c) $1\frac{8}{100}$ = 1.08
   (d) $16\frac{10}{100}$ = 16.10
   (e) $\frac{5}{100}$ = .05
   (f) $2\frac{1}{10}$ = 2.1

WRITE A NUMBER SENTENCE FOR EACH PROBLEM AND SOLVE:

3. While traveling the mountain roads of San Diego County, Mr. Lopez' car averaged 11.9 miles per gallon. When traveling on level city streets, his car averaged 15.5 miles per gallon. How many more miles per gallon did he average in the city than in the mountains?

$15.5 - 11.9 = 3.6$

4. During the months of January, February, and March, the weather bureau at Lindbergh Field recorded the following amounts of rainfall: January, .75; February, .63; March, .18. What was the total rainfall for the three-month period?

$.75 + .63 + .18 = 1.56$

USE THE FOLLOWING CHART TO SOLVE THIS PROBLEM:

5.

Rainfall in Inches		
Month	Actual	Normal
April	5.12	3.07
May	4.31	2.14
June	2.10	1.16
July	1.07	.21

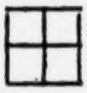
(a) What is the difference between the actual rainfall and the normal rainfall for:
April 2.05 ; May 2.17
June .94 ; July .86

(b) What is the total actual rainfall for the four-month period? 12.60 The total normal rainfall? 6.58

(c) What is the difference between the two amounts during the four-month period? 6.02

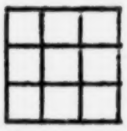
FIND THE NUMBER OF SQUARE SHAPES IN EACH EXAMPLE:

6. (a)



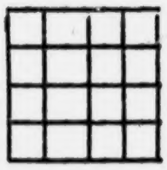
5

(b)



14

(c)



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CHAPTER V

GEOMETRY

For more than 4,000 years men have studied geometry for the purpose of trying to understand better the world in which they live. It is an extensive and important branch of mathematics which proceeds from familiar and concrete concepts toward more abstract and generalized concepts.

In this chapter, geometry will not be associated with the process of measurement (computing perimeters, areas, and volumes associated with certain geometric figures) nor will it be treated as a deductive science (building a series of theories). Instead, attention will be directed toward an intuitive approach (discovering that geometry can be developed by pupils from their own observations about shapes and the relations between shapes in the physical world around them). Vocabulary will be introduced where it is needed to help express ideas.

The pupils will learn about points, lines, line segments, planes, rays and angles. Moreover, the pupils will be introduced to models which represent the above ideas.

A. Concept of Point

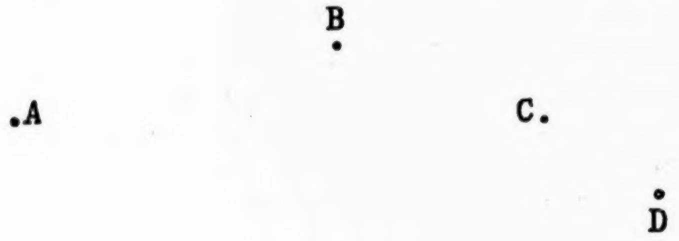
Objective. To introduce pupils to the geometric concept of point which is described as a precise, fixed location in geometric space; it has no physical length, width, or thickness.

Background Information for the Teacher. In geometry no attempt is made to accurately define the term point. Instead, it is described in terms of its properties. For example, in constructing physical models of geometric figures, dots and lines are drawn on paper or on the chalkboard. Just as a number is an idea physically represented by a numeral, so is a geometric point an idea physically represented by a dot or a set of dots. The dot we draw is a convenient approximation; no matter how small we make our dot, it still has length and width. Thus, a dot made on a sheet of paper is merely an attempt to mark the idealized geometric idea, the "point."

Geometric space, which is also undefined, can be thought of as an infinite set of points. Geometric space and geometric points, like numbers, are "ideas" which exist in the mind and are represented by symbols. When we say that space is an infinite set of points (immovable or fixed), we mean that the entire physical universe is completely "filled," or dense, with these points. There is no object in the universe, no matter how small or how distant, that is not "filled" with geometric points.

Geometric points do not move. For example, the set of points occupied by a pencil on a pupil's desk are fixed (immovable). If the pencil is moved to another location on the same desk, the pencil occupies another set of points or location in space. The original set of points (location) occupied by the pencil before it was moved has not moved; it is still there even though the pencil now occupies a new set of points.

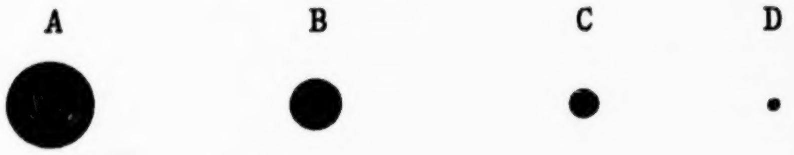
Since a point is usually represented by a dot, it is a practice to assign a capital letter to a point and thereafter to say "the point A" or "the point B." The letter assigned to a particular point is written next to the symbol (dot) which represents the point, as in the following example:



TEACHING PROCEDURE

Exploration. Indicate to the pupils that the mathematician is interested in exact locations. Locations of this kind are considered as points, represented by the symbol dot.

Draw or construct several dots on the chalkboard and label each as shown:



Ask the following questions:

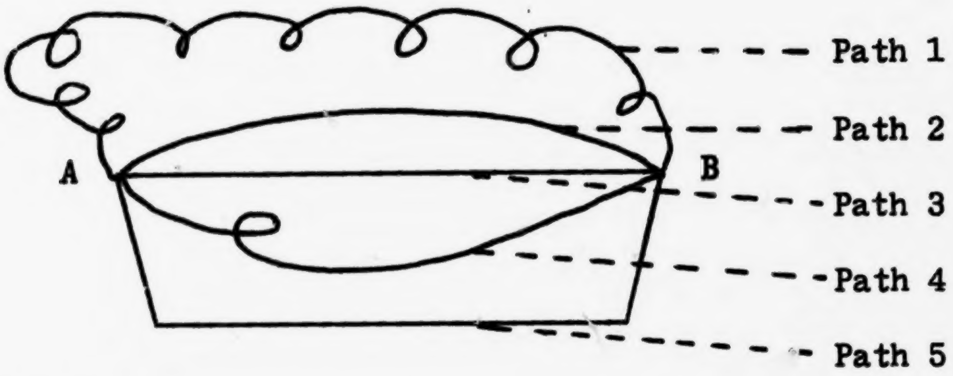
(1) "Is it possible to count the number of dots? Yes, 4 Is it possible to count the points represented by the dots? No, because dots are pictures; points are ideas which cannot be seen.

(2) "What is the advantages of being able to name a picture which represents a point?" To distinguish it from other points.

(3) "Which of the above dots best represents a point?" D; it is the smallest yet large enough to be seen.

(4) "If the chalk mark labeled D is erased, does the point move?" No. points are exact locations in space; they do not move.

Next, on the chalkboard, select two points and label one A and the other B. Indicate to the pupils that if we take points A and B in space and think of moving the tip of a piece of chalk from the location of A to the location of B, the path traced by the chalk would consist of a set of points. There are, of course, many paths which might be taken, for example:



Each of the five paths represented in the diagram are made up of different sets of points.

Ask the following questions:

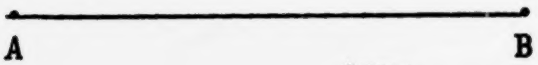
(1) "Which path represents the use of the least number of points? Path 3 Which path represents the shortest distance between point A and point B?" Path 3

(2) "Which path represents the use of the greatest number of points? Path 1 Which path represents the longest distance between point A and point B?" Path 1

B. Line Segment

Objective. To develop the concept of a line segment; a set of points which represents the shortest distance between two given points.

Background Information for the Teacher. The shortest or most direct path between two given points is called a line segment. A line segment has two end points, a first (or initial) point and a last (or terminal) point. Consequently, a line segment has a definite length. The illustration below represents a line segment.



The symbol for the "line segment AB" is $\overline{AB}$. It is read "line segment AB." Another name for the line segment AB is $\overline{BA}$ since both symbols represent the same set of points; thus, $\overline{AB} = \overline{BA}$.

A tightly stretched string (of a given length) is a good physical representation of a line segment. Other physical representations are: the edges of a book, the edge of a chalkboard tray, the edge of a desk, etc.

TEACHING PROCEDURE

Exploration. Draw three dots on the chalkboard and label them A, B, and C.

A. .B

C.

Tell the class that point A represents Alan's house, point B represents Bob's house, and point C represents Carl's house. Ask someone to show the shortest path from:

(1) Alan's house to Bob's house. A to B
(2) Bob's house to Carl's house. B to C
(3) Carl's house to Alan's house. C to A

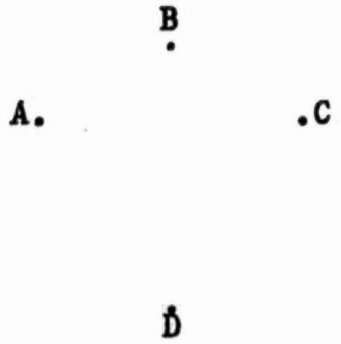
Explain to the class that in geometry the shortest path connecting two points is called a line segment. The points connected by a line segment are called the end points of the segment. Lastly, introduce them to the symbol for a line segment; indicate that a line segment is named in terms of its end points.

Ask the following questions:

(1) "What names might be given to the line segment connecting the points representing Alan's house and Bob's house?" $\overline{AB}$ or $\overline{BA}$.

(2) "Name other line segments which connect the points representing Bob's house and Carl's house. $\overline{BC}$ or $\overline{CB}$. Carl's house and Alan's house." $\overline{CA}$ or $\overline{AC}$

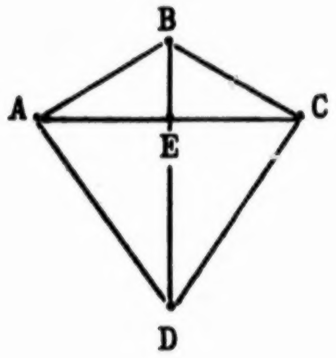
Next, on the chalkboard, draw four dots as shown below:



Before having any of the pupils come to the chalkboard to connect the points, ask the following:

(1) "How many line segments can be represented? 6 Name them." $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, $\overline{AC}$, and $\overline{BD}$

(2) "If we connect the points and include point E (see illustration below), how many line segments can we name? 10 Name them." $\overline{AB}$, $\overline{BC}$ $\overline{CD}$, $\overline{AD}$, $\overline{AC}$, $\overline{BD}$, $\overline{AE}$, $\overline{EC}$, $\overline{BE}$, $\overline{ED}$



At this time emphasize that a drawing is not always necessary. For example, ask the pupils to consider the path that light from a star follows to reach our eye.

(1) "Is this a representation of a line segment?" Yes

(2) "What are the representations of the end points?" The star and our eye

SUMMARY OF BASIC IDEAS

(1) A geometric point is undefined; it is described as an exact position or location in space.

(2) A geometric point does not move, it cannot be seen, and it represents an idea.

(3) A geometric point is represented by a symbol, dot, and labeled with a letter name.

(4) A line segment is a set of points which represents the shortest distance between two points.

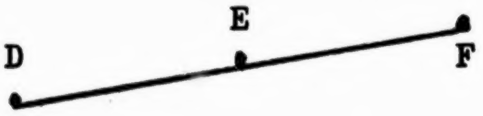
Pupil Exercises. Provide each pupil with worksheet No. 27.

All pupils should be provided with a ruler or straightedge to be used with example 6.

Encourage pupils to investigate the relationship between examples 4 and 5 and examples 4 and 6.

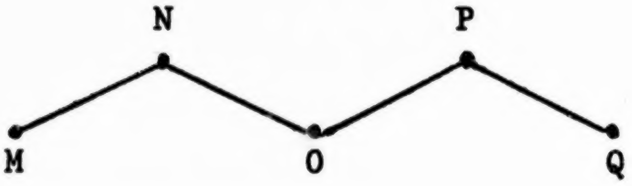
NAME ALL OF THE LINE SEGMENTS REPRESENTED IN EACH OF THE FOLLOWING:

1.



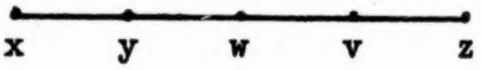
$\overline{DE}$ $\overline{EF}$ $\overline{DF}$

2.



$\overline{MN}$ $\overline{NO}$ $\overline{OP}$ $\overline{PQ}$

3.



$\overline{xy}$ $\overline{yw}$ $\overline{wv}$ $\overline{vz}$ $\overline{xW}$
$\overline{xV}$ $\overline{XZ}$ $\overline{yv}$ $\overline{yz}$ $\overline{wz}$

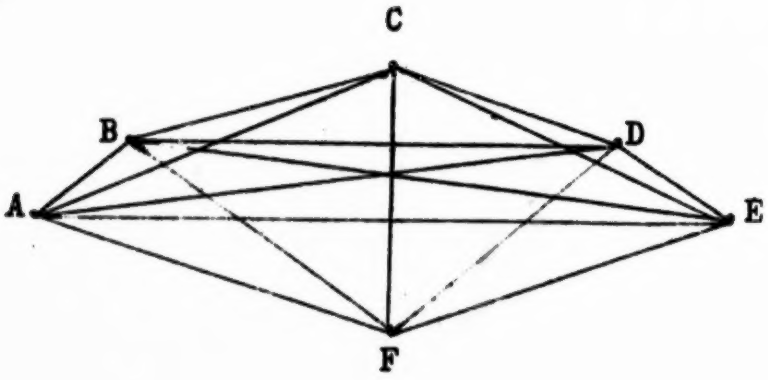
CAN YOU DISCOVER A PATTERN BY FILLING IN THE FOLLOWING:

4.

Number of end points	Total number of line segments that can be drawn	Pattern
2	1	1 = 1
3	3	3 = 1 + 2
4	6	6 = 1 + 2 + 3
5	10	10 = 1 + 2 + 3 + 4
6	15	15 = 1 + 2 + 3 + 4 + 5
7	21	21 = 1 + 2 + 3 + 4 + 5 + 6

5. How many line segments can be drawn using 9 given end points? 36 Show the pattern. 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36

6. How many different line segments can be drawn between 6 points shown below? (One way to find out is to draw them and count them.) 15 Check the patterns above.



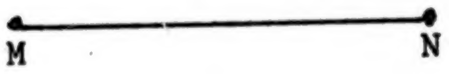
C. Lines

Objective. To develop the concept of a line; that a line is endless and has no end points.

Background Information for the Teacher. Most pupils probably have been calling a pencil mark drawn by a pencil and a ruler a line. However, the idea of a line as expressed on this page may be new to them. Some may not accept the fact that a line is an idea and that it cannot be seen but it can be represented graphically.

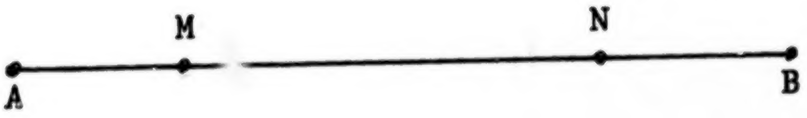
The following illustration (Figure 1) represents a line segment MN; it is the shortest distance between points M and N.

Figure 1



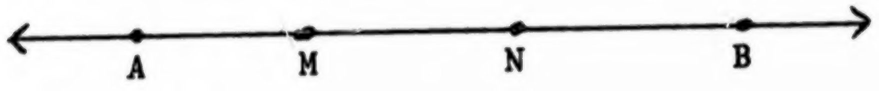
If we draw another line segment AB which will contain $\overline{MN}$ (see illustration Figure 2), we find that all of the points on $\overline{MN}$ are also on $\overline{AB}$ but not all of the points on $\overline{AB}$ are on $\overline{MN}$.

Figure 2



Since there is no limit to the number of line segments that can contai $\overline{MN}$, it becomes evident that we could go on finding longer and longer line segments that could contain $\overline{MN}$. Thus, all of these longer and longer line segments, if extended in both directions endlessly, will result in a straig line. Its symbol is $\overleftrightarrow{AB}$ and is read "the line AB." For brevity, a straigh line is called simply a line. Note that a line is a set of points, a particular set of points whose properties we describe but which cannot wholly be represented in a figure because of its indefinite length. Thus, we draw arrowheads at each end of the line to show that the line goes on and on endlessly in both directions. For example, the following illustrati (Figure 3) represent $\overleftrightarrow{AB}$ or "the line AB."

Figure 3



Since we can have a line with any two points on the line, the followin are the possibilities for Figure 3:

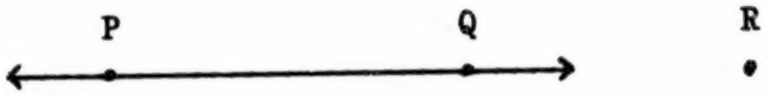
$\overleftrightarrow{AB}$, $\overleftrightarrow{AM}$, $\overleftrightarrow{AN}$, $\overleftrightarrow{MB}$, $\overleftrightarrow{MN}$, and $\overleftrightarrow{NB}$

TEACHING PROCEDURE

Exploration. Discuss the fact that when we say "line" in everyday speech, we are usually referring to a line segment. Emphasize that in geometry a line is endless.

On the chalkboard, draw the following illustration:

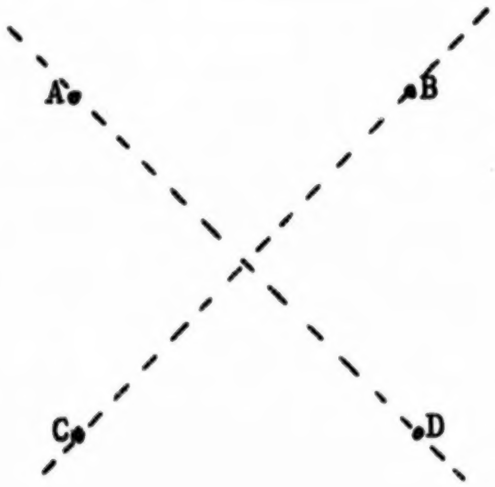
Figure 4



Ask, "Does point R lie on line segment PQ? No May point R lie on line PQ?" Yes, since a line may be extended in either direction; R may be on the extension to the right.

Next, on the chalkboard, draw four dots representing points arranged in a square (Figure 5).

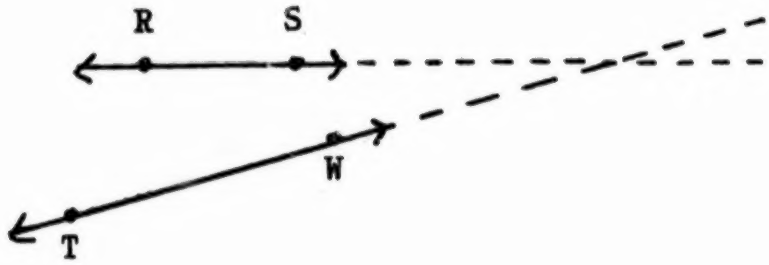
Figure 5



Ask, "If point A and point D represented a line AD and points B and C represented a line BC, would the two lines cross?"* Yes, because lines continue on and on without end.

Next, draw two lines such as $\overline{RS}$ and $\overline{TW}$ shown below (Figure 6):

Figure 6



Ask, "Would $\overleftrightarrow{RS}$ and $\overleftrightarrow{TW}$, as shown above, intersect in a point?"* Yes

Represent a point on the chalkboard and label it A. Select pupils to carry out the following activities (Figure 7):

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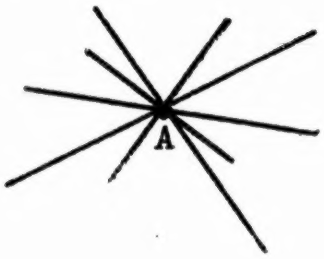
*Show the dotted lines after the pupils respond correctly.

Figure 7



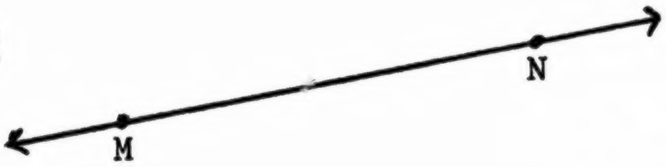
(a) Draw a line through point A.
(b) Draw a different line through point A.
(c) Draw three more lines through point A.

Ask the following questions:

(1) "How many lines can be drawn through one point?" More than can be counted

(2) "How many line segments can be drawn using points M and N in the illustration below (Figure 8)? One How many lines can be drawn that pass through both point M and point N? One How many points are needed to determine a line segment? Two How many points are needed to determine a line?" Two

Figure 8



SUMMARY OF BASIC IDEAS

(1) Through one point, any number of lines can be drawn.

(2) A line extends without end in both directions.

(3) If two distinct lines are given which intersect, their intersection consists of exactly one point.

(4) If two distinct points are given, there is exactly one line which contains both points.

D. Plane

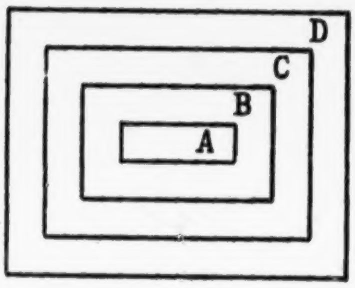
Objective. To development the concept of a plane by describing its properties and its relationship to lines and points.

Background Information for the Teacher. Any flat surface, such as the classroom wall, the floor, the top of a desk, or a sheet of paper, suggests the idea of a mathematical plane. A plane is not defined; it is described in terms of its properties and its relationship to lines and points.

Like our geometric line, a plane is unlimited in extent; that is, any flat surface used to represent a plane is only a representation of part of the plane. Moreover, we can only see models of a plane just as we see models of points and lines. We do not see a plane; we see only its representation.

The illustration below (Figure 1) shows the unlimited extent of a plane. The drawing illustrated by A continues to extend in all directions. Each larger quadrilateral* B, C, D, etc., represents the extension of A.

Figure 1



When studying planes, just as when studying lines, it is important to suggest representations in vertical and slanting, as well as in horizontal, positions. Emphasize also that the geometric plane has no thickness.

As shown in Figure 1, the set of points represented by plane A lie within plane B; the set of points represented by plane B lie within plane C; etc. In other words, A is a subset of B; A and B are subsets of C; etc.

TEACHING PROCEDURE

Exploration. Distribute a sheet of paper to each pupil. Emphasize that a flat sheet of paper may represent a plane. A plane is a set of points that can be extended endlessly into space. A plane has no thickness.

Next, have the pupils mark two points, A and B, on the piece of paper which represents a plane, M. Draw a line that passes through points A and B. Ask, "Are points A and B in the plane?" Yes Mark a point C on the line AB. "Is point C in plane M? Yes If we marked more points on the line, would these points be in the plane? Yes Do you think all the points on the line are in the plane?" Yes

---

*Quadrilateral is any closed plane geometric figure having four sides and four interior angles. Parallelograms, squares, rectangles, etc. are examples of quadrilaterals.

To show that there are an unlimited number of points in a plane, have the pupils mark a point D in the plane that is not on $\overline{AB}$. Next, draw lines through A and D; through B and D; through C and D.

Ask the following questions:

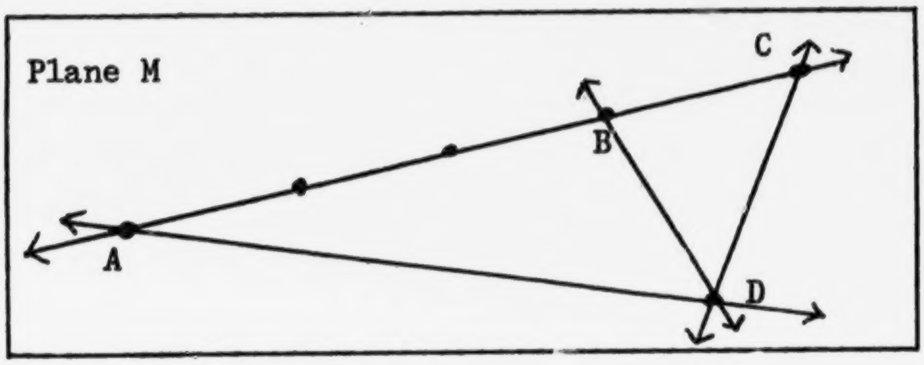
(1) "Are all points on $\overleftrightarrow{AB}$ contained in the plane?" Yes

(2) "Are all the points on $\overleftrightarrow{AD}$, $\overleftrightarrow{BD}$, and $\overleftrightarrow{CD}$ also contained in the plane?" Yes

(3) "If we marked more points on the paper, would all the points be contained in the plane? Yes Could we always mark more points on the paper? Yes Can we say that there are an unlimited set (infinite set), or number, of points in a plane?" Yes

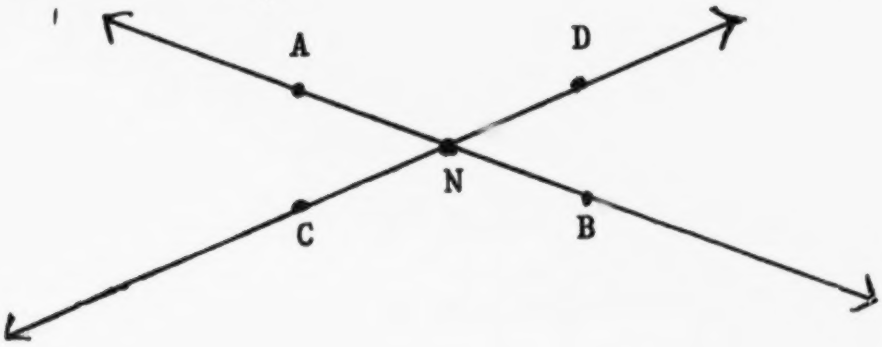
When all directions have been followed, the representative illustration (Figure 2) should look like this:*

Figure 2



To develop the concept of intersection, show the following illustration (Figure 3) on the chalkboard:

Figure 3



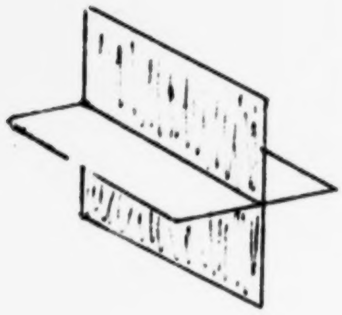
Ask, "Look at the illustration. Do the two lines AB and CD intersect at more than one point? No Is the point labeled N the point of intersection?" Yes

---

*The teacher may find it helpful to illustrate each step posed by the questions by using the chalkboard.

Next, on the chalkboard, represent two planes that cross each other as in the illustration (figure 4).*

Figure 4



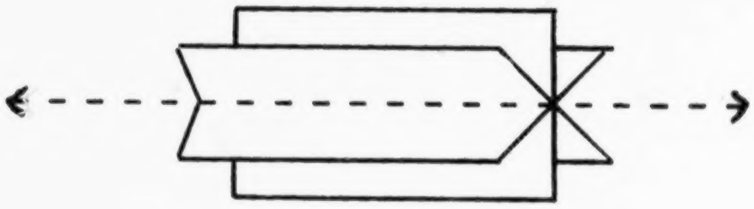
Ask, "Observe where the two planes intersect? Does the intersection suggest a line?" Yes. The line where two planes cross is called the line of intersection of two planes. "Do you think two planes can intersect in more than one line? No. In our classroom, where is the line of intersection of the plane represented by the floor and the plane of the wall?" Where the plane of the floor and the plane of the wall meet.

Using a model constructed of chipboard as in the illustration (figure 5), ask the following questions:

(1) "How many planes pass through the line of intersection? 3 Do you see that three planes can pass through one line?" Yes

(2) "Can four planes pass through a line of intersection? yes Can more than four planes pass through a line? Yes Do you think that an unlimited number of planes can pass through a line?" yes

Figure 5



(3) "Can we think of other examples which represent intersecting planes?"

(a) The planes of a paddle wheel boat.

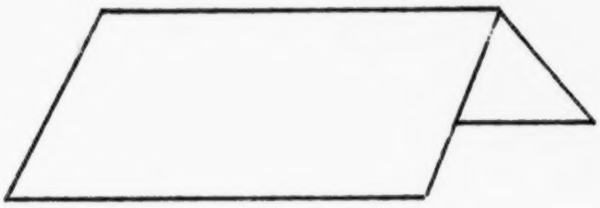
(b) Revolving door in a building.

---

*The pupils can make a model of two intersecting planes by using chipboard. Cut a slit in each of two pieces of chipboard (slit should be made up to the middle), insert one into the other, and open them up.

(c) Spine of a book.

(d) Folding a piece of paper so that it looks like this:



SUMMARY OF BASIC IDEAS

(1) A plane is a set of points.

(2) Like a line, a plane is unlimited in extent. A plane goes on and on.

(3) A plane has no thickness.

(4) If two different points lie in a plane, the line determined by the points lies in the plane.

(5) When two or more planes intersect, their intersection is a line.

E. Ray

Objective. To develop the idea of a "ray" and its relationship to a line and a line segment.

Background Information for the Teacher. The line determined by two different points such as C and D, (Figure 1), is called "the line CD" and represented by the symbol $\overleftrightarrow{CD}$. This line continues in both directions indefinitely. In Figure 2, the line segment CD, represented by the symbol $\overline{CD}$, does not continue in both directions. This segment has a definite starting point and end point and does not go beyond point C and point D.

Figure 1

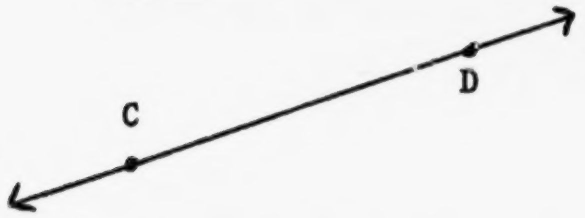
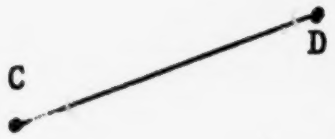
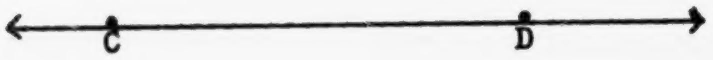


Figure 2



Besides $\overleftrightarrow{CD}$ and $\overline{CD}$, the two points C and D (Figure 3) determine another set of points called "the ray AB" and represented by the symbol $\overrightarrow{CD}$.

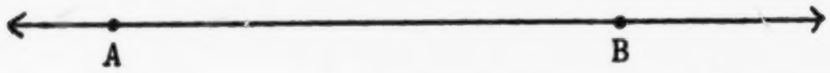
Figure 3



For example, the ray $\overrightarrow{CD}$ consists of the end point C and all those points to the right of point C, including point D and continuing indefinitely. To illustrate a ray, a beam of light emanating from a flashlight may serve as a model. Also, a ray of sun is a ray according to this definition if it is assumed that the ray of sun starts from a point. In looking at the illustration (Figure 3), we have another ray, that is $\overrightarrow{DC}$; in this case, D is the end point and the ray includes point D and all those points to the left of point D including point C and continuing indefinitely.

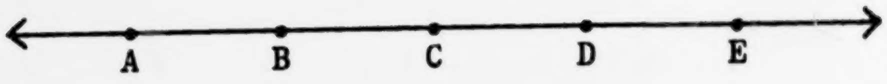
It is important to indicate that when naming lines or line segments, the order in which the letters are written is not important; however, order is important when naming a ray. The end point must be named first. In the illustration below (Figure 4), $\overrightarrow{AB}$ is quite different from $\overrightarrow{BA}$ but $\overleftrightarrow{AB}$ and $\overleftrightarrow{BA}$ are two names for the same line while $\overline{AB}$ and $\overline{BA}$ are two names for the same line segment.

Figure 4



Indicate that since a line is a set of points and a ray is part of a line, a ray is also a set of points. Furthermore, it cannot be seen, it has one end point, and since any point of the line may serve as the end point of a ray, a line contains more rays than can be counted. In the illustration (Figure 5), we have the following: $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$, $\overrightarrow{BC}$, $\overrightarrow{BD}$, etc; $\overrightarrow{ED}$, $\overrightarrow{EC}$, $\overrightarrow{EB}$, $\overrightarrow{EA}$, $\overrightarrow{DC}$, $\overrightarrow{DB}$, etc.

Figure 5



A ray may be a less familiar concept than a point, a line, or a line segment, but it is a very useful concept, particularly when dealing with angles.

TEACHING PROCEDURE

Exploration. If the conditions of your classroom permit a demonstration, darken your room and use a flashlight to introduce the idea of a ray. Indicate that the face of the flashlight is the point from which light emanates (end point) and continues in a given direction. Next, ask the pupils to think of other models of rays. They may suggest (a) a ray of sunshine, (b) a ray from a comic strip toy gun, (c) a ray from a filmstrip or motion picture projection, etc.

On the chalkboard, construct the following (Figure 1 and Figure 2) indicating that each figure represents part of a line called a ray.

Figure 1

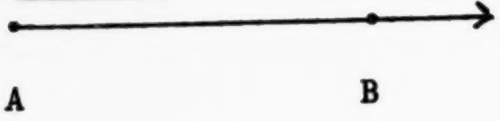
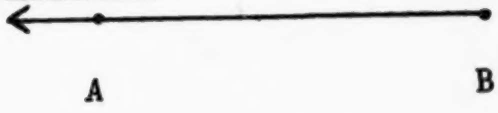


Figure 2



Ask the following questions:

(1) "What logical reasons can we think of for naming Figure 1 and Figure 2 as rays?" *

(a) Each has a beginning and no end.

(b) Figure 1 starts at point A and continues through point B; Figure 2 starts at point B and continues through point A.

(c) Each figure indicates the direction of the set of points.

(2) "If we represent Figure 1 as $\overrightarrow{AB}$ (read as 'ray AB') and Figure 2 as $\overleftarrow{BA}$ (read as 'ray BA'), can we say that the order does or does not make a difference when naming a ray? Why?" The order does make a difference when naming a ray because two different sets of points are involved.

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* Answers will vary.

F. Angle

Objectives. (1) To develop the idea of an "angle" which is a geometric figure or a set of points. (2) To emphasize that an angle is the union of two rays which have the same end point but which are not parts of the same line.

Background Information for the Teacher. An angle is made up of two rays that have the same end point. For example, Figure 1, Figure 2, and Figure 3 each illustrate the idea of an angle.

Figure 1

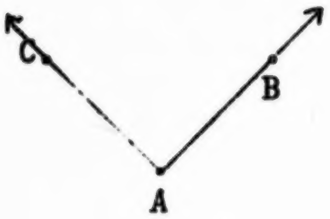


Figure 2

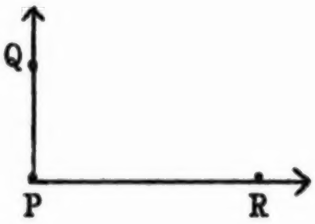
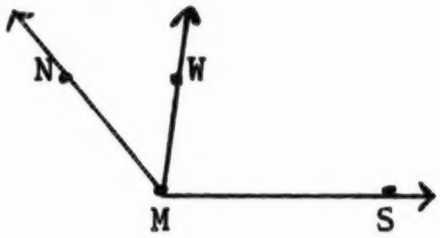


Figure 3



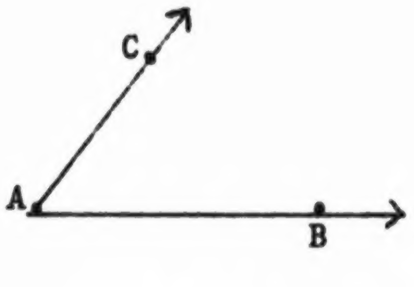
In Figure 1, the angle is formed by the union (joining) of $\overrightarrow{AB}$ and $\overrightarrow{AC}$; each ray has a common end point, A. The common end point at which the rays meet is called the vertex of the angle. Point B in Figure 1 is not a part of the $\overrightarrow{AC}$; however, it is part of the angle. In Figure 2, P is the vertex; it is the end point of $\overrightarrow{PR}$ and $\overrightarrow{PQ}$. Figure 3 illustrates three angles; $\overrightarrow{MN}$ and $\overrightarrow{MW}$ form one angle with M as the vertex. $\overrightarrow{MW}$ and $\overrightarrow{MS}$ form the second angle; M is the vertex. The third angle is formed by $\overrightarrow{MN}$ and $\overrightarrow{MS}$; M is the vertex.

In naming angles, three letters which name points of the angle are used. The letter which names the vertex is always named second or is in the middle. For example, the angle in Figure 1 may be named as BAC or as CAB; the vertex is in between B and C. Figure 2 may be named as "angle QPR" or "angle RPQ"; the vertex is in the middle. However, in Figure 3, each angle must be named separately: NMW or WMN; WMS or SMW; and NMS or SMN.

TEACHING PROCEDURE

Exploration. Direct the attention of the pupils to the following illustration (Figure 1) which may be constructed on the chalkboard:

Figure 1



Ask, "How many rays are shown in this illustration? two Name each ray. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ What is the common end point for each ray?" A

Indicate to the pupils that when two rays have the same end point, the geometric figure formed is called an "angle." Construct angles of varying sizes and report the identification of the rays and common end point.

Next, ask the pupils to point out some physical examples in the classroom or near the school building which represent angles. Some of the responses might be:

(a) Cross bars of a telephone pole.

(b) Edges of a book which meet at one point.

(c) Edges of a desk which meet at one point.

(d) Folded pieces of paper with a common end point.

(e) A partially open compass with the fulcrum representing an end point.

Pupil Exercises. Provide each pupil with worksheet No. 28.

With the more able pupils, encourage independent work, especially with examples 8 and 9. However, with the less able pupils, read and interpret the instructions, making sure they understand what they are to do.

Interim Mathematics Unit - Grade 6
San Diego City Schools (1-65)

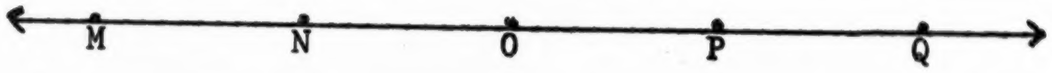
Worksheet No. 28

WRITE THE WORD "true" OR "false" IN EACH BLANK SPACE.

1. A line has one end point. false
2. A line has two end points. false
3. A line has no end points. true
4. If two different lines in space intersect, their intersection is one point. true
5. If two different planes intersect, their intersection is a line. true

COMPLETE EACH OF THE FOLLOWING:

6. One of the line segments shown in the drawing is MN. Write the names of nine more line segments shown on this line.



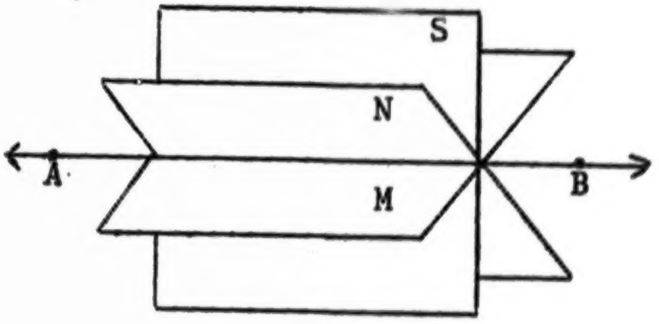
$\overline{MO}$ $\overline{MP}$ $\overline{MQ}$ $\overline{NO}$ $\overline{NP}$ $\overline{NQ}$ $\overline{OP}$ $\overline{OQ}$ $\overline{PQ}$

7. In the illustration, what is the intersection of:

(a) Plane M and plane N $\overleftrightarrow{AB}$

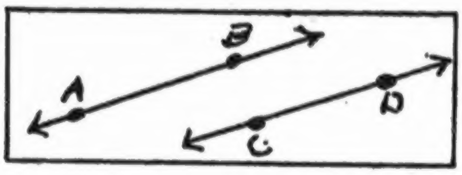
(b) Plane N and plane S $\overleftrightarrow{AB}$

(c) Plane M and plane S $\overleftrightarrow{AB}$

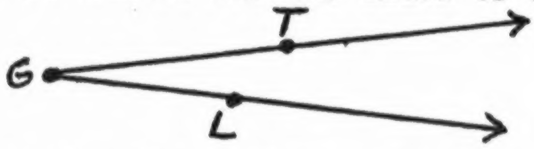


8. Suppose there are two different lines on a plane. Must they always intersect?

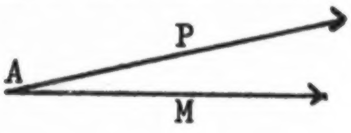
No Give an example.



9. Suppose $\overrightarrow{GT}$ and $\overrightarrow{GL}$ are the sides of an angle. Draw the angle.



10. Two names for the following angle are: angle PAM and angle MAP



GLOSSARY

ADDEND. The name given to any of the numbers to be added.

ADDITION. The process of joining two or more numbers and determining the total quantity. It is one of the four fundamental operations (processes) of arithmetic. Through addition we can find a sum without counting.

ALGORITHM.(Algorism). Any of many short-cut computational schemes or procedures in arithmetic, especially of those which require various, different or repeated steps. More generally, this term is applied to the form in which arithmetical operations are expressed. Thus $2\overline{)6}$ denotes the division algorism. The properties of arithmetic can be used to justify the procedures used in an algorithm.

ARRAY. An orderly arrangement of objects in rows and columns.

ASSOCIATIVE PRINCIPLE OF ADDITION AND MULTIPLICATION. Fundamental structural properties of elementary arithmetic referring to the order in which the operations of addition and multiplication are to be carried out when more than two numbers are involved. Examples are: $(3+4)+2 =$ $3 + (4 + 2)$ and $(6 \times 2) \times 3 = 6 \times (2 \times 3)$. To generalize: $(a+b) + c =$ $a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$.

BASE TEN. A system of numeration or a place-value system using ten as its fundamental counting group or base.

BINARY OPERATION. An operation which assigns to a pair of numbers (an order pair) another number which is called the sum in addition; the product in multiplication, etc.

CHECK. To verify the correctness of an answer or solution. It is not to be confused with "proof." When one checks the correctness of a sum or product by performing the operation in reverse order, one does not prove the result but merely seeks verification or added evidence for correctness.

COLUMN. A vertical arrangement of things, numerals, or any other symbols--most commonly used in "column addition." It is also the term which applies to the vertical arrangements in operations with arrays and tables, such as addition table, subtraction table, etc.

COLUMN ADDITION. A vertical arrangement of numerals for the purpose of facilitating the addition of more than two numbers. It is essential that the beginning learner become well acquainted with both the horizontal and vertical operational arrangements.

COMMUTATIVE PRINCIPLES OF ADDITION AND MULTIPLICATION. These are fundamental structural properties of elementary arithmetic which refer to the order in which the operations of addition and multiplication can or cannot be performed when only two numbers are involved. Examples are: $4 + 6 = 6 + 4$ and $3 \times 5 = 5 \times 3$. To generalize: $a + b = b + a$ and $a \times b = b \times a$.

CONCRETE. Tangible objects used to make an arithmetic situation more meaningful; the antithesis of abstract.

DIGIT. The basic numerals used in place-value systems of numeration. In the decimal system (base ten) the digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The numeral 251 has three digits.

DISTRIBUTIVE PRINCIPLE OF MULTIPLICATION OVER ADDITION. One of the fundamental structural properties of arithmetic. An example is: $3 \times (4 + 3) = (3 \times 4) + (3 \times 3) = 12 + 9 = 21$. To generalize: $a \times (b + c) = (a \times b) + (a \times c)$, where a, b, and c are any numbers. This principle differs from the commutative and associative principles of multiplication in that it involves two of the four basic operations, that is, multiplication and addition.

DIVIDE. To divide one number by another means to find out how many times the second number can be subtracted from the first. For example, in $12 \div 4$, the procedure is to subtract 4 from 12 as many times as possible and then state as the quotient the number of times that the subtraction could be performed. It is the inverse of multiplication.

DIVISOR. The number in the division process by which another number is to be divided. In $15 \div 3 = 5$, the 3 is the divisor.

EQUAL VALUE. Two or more arithmetical expressions having the same numerical value though they may have very different forms.

EQUALITY. The general property indicated by two or more statements indicating equal value.

EQUIVALENCE. A relation that holds between two sets that can be put into one-to-one correspondence. For example: the set of days in one week is equivalent to the set of bright stars in the Big Dipper. Each set has seven members.

EXPANDED NOTATION. A numeral that shows the number represented by each digit in a place-value numeral. For example, the expanded notation for the numeral 379 is: $300 + 70 + 9$, or $(3 \times 100) + (7 \times 10) + (9 \times 1)$.

FACTOR. Any of the numbers to be multiplied to form a product. Thus, in $3 \times 4 = 12$, the 3 and 4 are factors of 12. For beginning learners it is wise to insist on "factors" or "factor" rather than on "multiplier" and "multiplicand."

IDENTITY ELEMENT. A number and its symbol which does not change the value when a certain operation is performed. Zero is called the identity element of addition; "one" is called the identity element of multiplication.

IDENTITY. An equation like $5 = 5$ in which the values on either side of the = sign are identically the same. A principle such as the commutative principle of addition, $a + b = b + a$, is an identity since it is true for all numbers a and b.

INTERSECT. To meet or cross in a point or confined space.

INVERSE (OF AN OPERATION). The term should be restricted in its elementary use to operations. Subtraction is the *inverse* of addition; division is the *inverse* of multiplication. For beginning learners, the idea of *operational inverse* may be conveyed by showing that one operation *does* what its inverse *undoes*.

LINE SEGMENT. A unit of definite length which can be described in appropriate units of measure.

MULTIPLE. A term indicating a product obtained by the multiplication of a specified number and any whole number. If the specified number is 4, for example, we speak of the series 4, 8, 12, 16, etc., and the "multiples of 4." The term "multiple" should replace, where possible, the term "multiplication facts."

MULTIPLICAND. Thinking of multiplication as repeated additions, that factor which is to be added a specified number of times is the multiplicand. Thus, in 4 × 5, the factor 5 is the multiplicand.

NUMERALS. Symbols used to denote numbers, as the Hindu-Arabic or the Roman numerals. Numerals are *concrete*; they can be written on the chalkboard. The numbers which they denote or for which they stand are *abstract*; they can only be *represented* by symbols.

OPERATION. In elementary arithmetic this refers to an "action" relation; the process of carrying-out the rules of a procedure or of an algorism. The four fundamental *operations* are: addition, subtraction, multiplication, and division.

PARTIAL PRODUCT. Literally, a part of a product but not the "whole product." It is used in elementary arithmetic with regard to the written algorithm of multiplication. In this algorithm each digit in the multiplier produces one partial product. The final product is then the sum of the partial products.

PLACEHOLDER. Any symbol (frame, letter, etc.) designed to hold a place in a number or mathematical expression. In the contemporary approach to elementary arithmetic, "placeholder equations" play an important role, especially with younger pupils.

PRINCIPLE. A property of numbers is a fact or characteristic that is always true for a specified situation involving numbers. (See commutative principles of addition and multiplication, associative principles of addition and multiplication, etc.). Synonym for property.

RENAMING. The process of replacing a numeral by a different numeral to name the same number. For example, the numeral 256 names a particular number: 250 + 6; 200 + 50 + 6; 200 + 40 + 16; and 16 × 16 also name 256.

REPEATED ADDITION: The "long form" of multiplication.

RESULT. The final outcome of an operation or algorithm. Whenever possible, one should insist on calling a result by its specific name, that is, "sum," "product," "missing addend," "quotient," etc.

ROW. A line or arrangement running horizontally. For example, the entries in a multiplication table which run from left-to-right are called "rows."